Special Relativity: Lecture 7

1 Force

Remember Lecture 4:

$$F_\mu = \frac{\mathrm{d}\,p_\mu}{\mathrm{d}\tau} = \gamma(u)(\mathbf{f},\ \frac{i}{c}\mathbf{u}\cdot\mathbf{f})$$

where $\mathbf{f} = \mathrm{d}\mathbf{p}/\mathrm{d}\,t$ is the relativistic 3-force.
Under LT ($S \to S'$):

$$F'_\mu = L_{\mu\nu}F_\nu$$

so using eq.(IV.3), viz. $\gamma(u') = \gamma(u)\gamma(v)(1 - \mathbf{v}\cdot\mathbf{u}/c^2)$:

$$\begin{aligned} f'_x &= \left(f_x - \frac{v\mathbf{u}\cdot\mathbf{f}}{c^2}\right)\Big/\left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right) \\ f'_y &= f_y/\gamma(v)(1 - \mathbf{v}\cdot\mathbf{u}/c^2) \\ f'_z &= f_z/\gamma(v)(1 - \mathbf{v}\cdot\mathbf{u}/c^2) \end{aligned} \qquad (VII.1)$$

Special Case
Particle of constant mass m_0 moving under the influence of a force acting in a fixed direction Ox.

- Let S' be the particle's rest frame at a given instant t. In S': $F'_\mu = (f', 0, 0, 0)$ since $u' = 0$ and $\gamma(0) = 1$.
- Apply the inverse LT ($F_\mu = L^T_{\mu\nu}F'_\nu$) to transform to a frame S in which particle has a velocity v (along Ox in S):

$$\begin{pmatrix} F_x \\ F_y \\ F_z \\ F_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} f' \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

so:

$$\left.\begin{aligned} F_x &= \gamma(v)f' && = \gamma(v)f_x \\ F_y &= 0 && = \gamma(v)f_y \\ F_z &= 0 && = \gamma(v)f_z \\ F_4 &= i\beta\gamma(v)f' && = i\gamma(v)vf_x/c \end{aligned}\right\}\ \text{in } S,$$

hence

$$f_x = f'; \qquad f_y = f_z = 0,$$

i.e. direction and magnitude of force are preserved under the LT. The equation of motion in S is

$$f_x = \frac{dp_x}{dt} = f' \qquad (VII.2)$$

f' is known as the *proper force* and is the space part of the 4-force in the instantaneous rest frame.

2 Acceleration

The 3-acceleration is $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$ in a given frame. Consider:

$$\begin{aligned} \mathbf{f} &= \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma(v)\mathbf{v}) \\ &= m_0\gamma(v)\mathbf{a} + m_0\mathbf{v}\frac{\mathrm{d}\gamma(v)}{\mathrm{d}t} \end{aligned} \qquad (VII.3)$$

so in general $\mathbf{f}$ and $\mathbf{a}$ are *not* in the same direction.
What is the form of the 4-acceleration?

$$\mathcal{A}_\mu = \frac{\mathrm{d}V_\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}V_\mu}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma\left\{\frac{\mathrm{d}(\gamma\mathbf{v})}{\mathrm{d}t}, ic\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\} \qquad (VII.4)$$

There are too many γ's and $\mathrm{d}\gamma/\mathrm{d}t$'s to make this an easy approach. To get the transformation equations for 3-accelerations between IFs S and S', it is easier to start from the velocity transformation eq.(IV.4) and differentiate ($\mathrm{d}/\mathrm{d}t'$ corresponds to S', $\mathrm{d}/\mathrm{d}t$ corresponds to S, and use $\mathrm{d}t'/\mathrm{d}t = \gamma(v)(1 - vu_x/c^2)$) to get:

$$\begin{aligned} a'_x &= \frac{a_x}{\gamma(v)^3\left(1 - vu_x/c^2\right)^3} \\ a'_y &= \left[a_y + \frac{u_y v/c^2}{(1 - u_x v/c^2)}a_x\right] \Big/ \gamma(v)^2\left(1 - \frac{vu_x}{c^2}\right)^2 \\ a'_z &= \left[a_z + u_z v/c^2/\left(1 - u_x v/c^2\right)a_x\right] \Big/ \gamma(v)^2\left(1 - \frac{vu_x}{c^2}\right)^2. \end{aligned} \qquad (VII.5)$$

Complicated! Constant acceleration in S will not necessarily be constant in S'.

3 Comoving approach

This offers considerable simplification and a consistent approach. Suppose at a given instant t the particle has a velocity v as measured in a fixed frame S. Apply a LT to frame S' moving with velocity v, in which the particle is *momentarily* at rest. In S' the particle velocity $\mathbf{u}' = 0$, and the acceleration is $\mathbf{a}'$.
Now apply the inverse LT to get back to S:

$$\begin{aligned} a_x &= a'_x/\gamma^3 \\ a_y &= a'_y/\gamma^2 \\ a_z &= a'_z/\gamma^2. \end{aligned} \qquad (VII.6)$$

At a later time $t + \Delta t$, the LT with velocity $v + \Delta v$ would be used in eq.(VII.6). Thus in the *fixed* frame S we can integrate the equation of motion. As an example consider the case of a constant force in S' — acting along Ox' to give acceleration $\mathbf{a}' = (\bar{a}, 0, 0)$. Then in S:

$$a_x = \frac{d^2x}{\mathrm{d}\,t^2} = \bar{a}\left(1 - \frac{v^2}{c^2}\right)^{3/2}. \qquad (VII.7)$$

If the particle is at rest at the origin in S at $t = 0$, eq.(VII.7) yields the trajectory

$$x = \frac{c^2}{\bar{a}}\left(1 + \frac{\bar{a}^2 t^2}{c^2}\right)^{1/2} - \frac{c^2}{\bar{a}}.$$

4 "Twin Paradox"

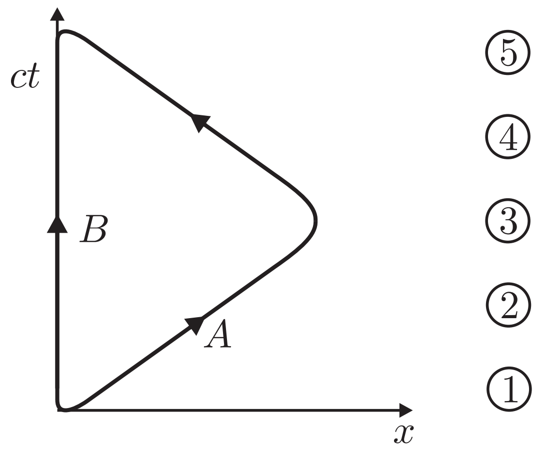
In his 1905 paper introducing special relativity Einstein states: "If one of two synchronous clocks at A is moved in a closed curve with constant velocity until it returns to A, the journey lasting t seconds then by the clock which has remained at rest the travelled clock on its arrival at A will be $\frac{1}{2}\frac{v^2}{c^2}t$ seconds slow."

This statement has caused controversy for years. Some say that it violates the postulates of special relativity, others argue that general relativity is needed because to complete a closed path the clock must be accelerated.

Suppose A and B are identical twins. A travels on a round trip to the star α-centauri and back, while B remains on Earth.

Consider A's journey:

1. acceleration to cruising velocity v
2. outward journey with vel. v
3. reversal of motion
4. return journey with vel. v
5. decceleration



One can imagine that the times for (1) and (5) are made identical and that the times for (1), (5) and (3) are very much less than the times for (2) and (4). In particular for the stages (2) and (4) when A has velocity v relative to B apply time dilation

$$\Delta t = (1 - v^2/c^2)^{1/2}\ \Delta t_A.$$

To be more exact apply the comoving approach:

$$\Delta t = \int_{\text{trajectory A}} (1 - v^2/c^2)^{1/2}\ \mathrm{d}\,t_A \qquad (VII.8)$$

So it is clear that on B's clock, A will be "younger". The "paradox" arises when one tries to argue that the principle of relativity should apply — no preferred frames — and the *same* answer should result if we consider A at rest and B undertaking the journey. Surely B will now appear to age less. Contradiction? Is time dilation a fiction or is it the acceleration — particularly stage 3 — which is responsible for the asymmetry. The last public flare-up was in 1972 in the pages of *Physics Today*. (See also Bowler p.54 and the usenet relativity FAQ at http://hepweb.rl.ac.uk/ppUK/PhysFAQ/twin_paradox.html for a resolution.)

We have already looked at experiments that confirm eq.(VII.8)—at least for unstable particles. If this doesn't convince then consider the experiment of Hafele & Keating (*Science* **177** (1972) 166). They used atomic clocks — one at rest on the Earth and the other flown around the world on commercial airliners. Their results were (all times in nsec):

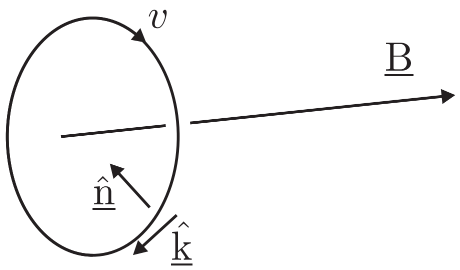
	East	West	
Prediction	144 ± 14	179 ± 18	Gravitational
	-184 ± 18	96 ± 10	Kinematic
	-40 ± 23	275 ± 21	Net
Measurement	-59 ± 10	273 ± 7	

A and B *are* different, A undergoes acceleration and would be able to measure it.

5 Another example of force

Consider the motion of a charged particle in a constant magnetic field, when the particles are injected $\perp$ to $\mathbf{B}$.

Assume for the moment that the Lorentz force is correct (see Appendix). In the frame in which the field $\mathbf{B}$ is fixed the equation of motion is:



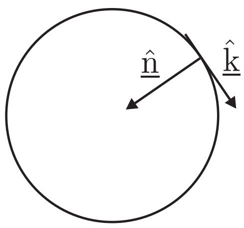
$$\mathbf{f} = q\mathbf{v} \wedge \mathbf{B}.$$

So $\mathbf{f} \cdot \mathbf{v} = 0$, but the rate of working is related to the rate of change of relativistic energy (see eq.(IV.13)) so $dE/\mathrm{d}t = 0$, but $E = mc^2\gamma$.

$$\text{Therefore} \quad \frac{\mathrm{d}\gamma}{\mathrm{d}t} = 0 \qquad \text{or} \quad (1 - v^2/c^2)^{-3/2}\frac{dv}{\mathrm{d}t} = 0.$$

Conclusion: v is constant, only its direction changes. Write $\mathbf{v} = v\hat{\mathbf{k}}$ and introduce a unit vector $\hat{\mathbf{n}}$ $\perp$ to $\mathbf{v}$ and $\mathbf{B}$. The equation of motion is then

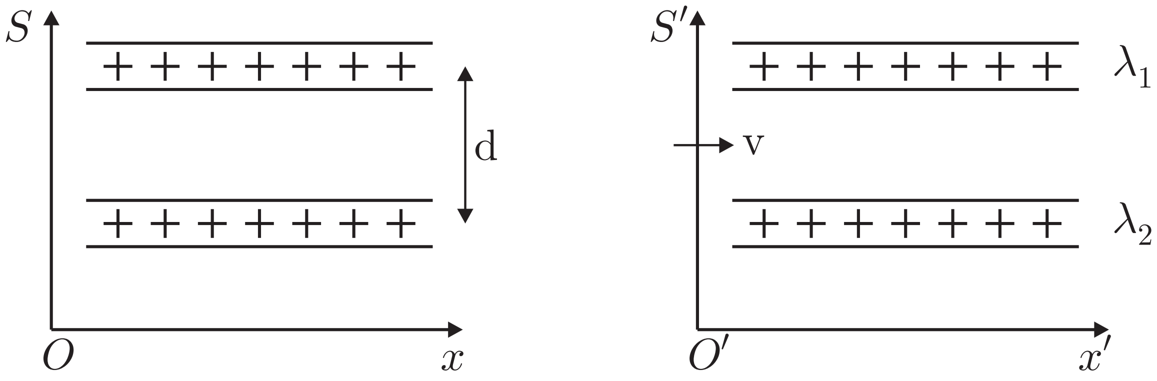
$$\begin{aligned} \mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= \frac{\mathrm{d}(m\gamma(v)\mathbf{v})}{\mathrm{d}t} = m\gamma v\frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t}, \\ \text{but} \quad \mathbf{f} &= qvB\hat{\mathbf{n}}, \\ \text{so} \quad \frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t} &= \frac{qB}{m\gamma}\hat{\mathbf{n}}. \end{aligned}$$



This is the equation of a rotation with frequency $\omega = qB/m\gamma$. This is the *cyclotron frequency* and is *not* constant because of the γ factor.

6 Magnetism as a relativistic effect

Consider two infinite line charges – linear charge densities λ_1 and λ_2.
In frame S' they are at rest.



Using Gauss, the field of λ_1 is $E'_1 = \lambda_1/2\pi\epsilon_0 d$ at the position of 2, hence the repulsive force is

$$F' = \frac{\lambda_1\lambda_2}{2\pi\epsilon_0 d} \qquad \text{per unit length.}$$

Consider the force on a length ℓ': $F'\ell'$. Now transform to frame S wrt which S' has velocity v along the wire direction. The force is transverse so $F = F'/\gamma$, and the longitudinal length is contracted $\ell = \ell'/\gamma$. So

$$F\ell = F'\ell'/\gamma^2 = \frac{\lambda_1\lambda_2}{2\pi\epsilon_0 d}(1 - v^2/c^2) \qquad (VII.9)$$

Now consider the same problem (in frame S) using conventional EM. We have the electrostatic *repulsion* as above: $F_{\text{elec}} = \dfrac{\lambda_1\lambda_2}{2\pi\epsilon_0 d}$. But there is also a magnetic *attraction* from the motion of the charges:

$$2\pi d B_1 = \mu_0\lambda_1 v \qquad \text{or} \qquad B_1 = \frac{\mu_0\lambda_1 v}{2\pi d}$$

$$\therefore \quad F_{\text{mag}} = \frac{\mu_0\lambda_1\lambda_2 v^2}{2\pi d}$$

$$\therefore \quad F_{total} = F_{\text{elec}} + F_{\text{mag}} = \frac{\lambda_1\lambda_2}{2\pi\epsilon_0 d}(1 - \epsilon_0\mu_0 v^2). \qquad (VII.10)$$

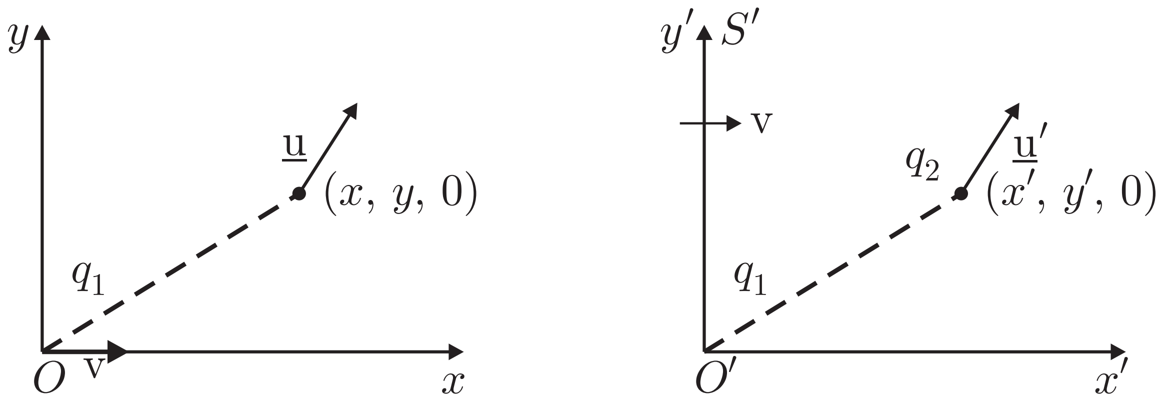
But $\epsilon_0\mu_0 = 1/c^2$ so eqs.(VII.9) and (VII.10) agree and we can view the magnetic term as a relativistic correction! Magnetic effects are only measurable because of the complete cancellation of electrostatic forces in neutral matter.

7 Appendix

Form of the Lorentz force from Coulomb's law and the LT for force.
Coulomb's Law:

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3} .$$

This is true in the rest frame of the source charge ($\mathbf{r}_1$) — independent of motion of test charge. Consider the standard IFs S and S' with the source charge q_1 at rest at the origin O' in S' and q_2 at $(x', y', 0)$ with velocity $\mathbf{u}'$. In S, q_1 will have a velocity v along Ox.



In S'

$$\begin{aligned} F'_x &= \frac{q_1 q_2}{4\pi\epsilon_0}\frac{x'}{r'^3} , \\ F'_y &= \frac{q_1 q_2}{4\pi\epsilon_0}\frac{y'}{r'^3} . \end{aligned}$$

$$(VII.11)$$

Now apply the inverse LT for 3-force between S' and S

$$\begin{aligned} F_x &= \{F'_x + \frac{v}{c^2}(\mathbf{F}' \cdot \mathbf{u}')\}/(1 + \frac{vu'_x}{c^2}) , \\ F_y &= \frac{F'_y}{\gamma(v)(1 + vu'_x/c^2)} . \end{aligned}$$

$$(VII.12)$$

The next step is to use the velocity transformation

$$u'_x = \frac{(u_x - v)}{(1 - vu_x/c^2)} \quad \text{and} \quad u'_y = \frac{u_y}{\gamma(1 - vu_x/c^2)} ,$$

to write the RHS of eq.(VII.12) in terms of unprimed quantities. After some effort, the result is

$$\begin{aligned} F_x &= \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\gamma}{r'^3}\left(x + \frac{v}{c^2} y u_y\right) , \\ F_y &= \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\gamma}{r'^3}\left(y - \frac{v}{c^2} y u_x\right) , \end{aligned}$$

$$(VII.13)$$

where

$$r'^2 = (\gamma x)^2 + y^2 \qquad (z = 0).$$

Now in IF S, define the electric and magnetic fields

$$\mathbf{E} = \frac{q_1}{4\pi\epsilon_0} \frac{\gamma \mathbf{r}}{r'^3} \ .$$

Now we can identify $q_2 E_x$ and $q_2 E_y$ in eq.(VII.13) and write

$$\mathbf{F}_{\text{elec}} = q_2 \mathbf{E}.$$

Now $\quad \mathbf{B} = \dfrac{1}{c^2} \mathbf{v} \wedge \mathbf{E}$,

$$\begin{aligned} \text{so} \qquad B_x &= \frac{1}{c^2}(v_y E_z - E_y v_z) = 0 \ , \\ B_y &= \frac{1}{c^2}(v_z E_x - v_x E_z) = 0 \ , \\ B_z &= \frac{1}{c^2}(v_x E_y - v_y E_x) = \frac{1}{c^2} v_x E_y \ , \end{aligned}$$

thus $\mathbf{u} \wedge \mathbf{B}$ gives

$$\begin{aligned} \mathbf{u} \wedge \mathbf{B}|_x &= \frac{u_y v_x E_y}{c^2} \ , \\ \mathbf{u} \wedge \mathbf{B}|_y &= -\frac{u_x v_x E_y}{c^2} \ , \end{aligned}$$

and

$$\mathbf{F}_{\text{mag}} = q_2 \mathbf{u} \wedge \mathbf{B}.$$

Thus the force on the moving charge in S is

$$\mathbf{F} = q_2(\mathbf{E} + \mathbf{u} \wedge \mathbf{B}). \tag{$VII.14$}$$

This is the Lorentz Force—we have shown that it is (at least plausible that the form is) a consequence of Coulomb's law and the Lorentz transformation.

There is a better way to do this and that is to use the covariant formalism.