Special Relativity: Lecture 1

1 Definitions

Inertial Frame of Reference (IF) is an *unaccelerated* frame of reference, in which Newton's laws hold. The standard example is a frame of reference in empty space moving with uniform velocity wrt the fixed stars. An IF moving with constant velocity wrt another IF is also an IF.

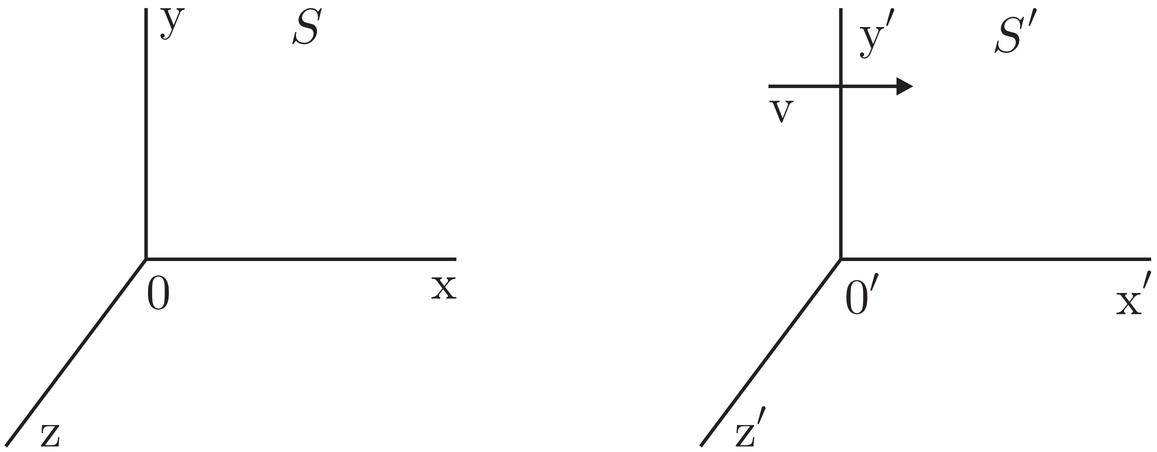
Invariant, a quantity that does not change its value under a transformation, e.g. the length of a vector under a coordinate rotation.

Covariant equations have the same *form* under a transformation, the physical content is the same, e.g. Newton's Laws of motion are covariant wrt a coordinate rotation.

Special Relativity deals with physical phenomena true in all IFs, e.g. mechanics and electromagnetism.

General Relativity deals with physical phenomena in *accelerated* frames of reference as well as IFs and is a theory of gravitation.

Standard IFs S and S' will be defined to be RH coordinate systems with S at rest and S' moving with a velocity $\mathbf{v}$ directed along the positive x-axis in S. We shall assume that S and S' coincide at $t = 0$ so that $t = t' = 0$ and $x = x' = 0$.



2 Galilean Relativity

Newton's laws of motion are covariant wrt coordinate rotations. They are also covariant wrt the Galilean transformation between IFs S and S' with constant relative velocity. For the standard IFs S and S' defined above, the Galilean transformation is:

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t.$$

By differentiation we have that;

$$\frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x'}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - v,$$

which gives the Newtonian law for relative velocity $u' = u - v$. Acceleration is unchanged since

$$\frac{\mathrm{d}^2 x'}{\mathrm{d}t'^2} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\,.$$

In Newtonian mechanics the mass of a body, m, is an invariant, hence

$$\mathbf{F} = m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} \;\text{ in } S,$$

becomes

$$\mathbf{F}' = m\frac{\mathrm{d}^2\mathbf{x}'}{\mathrm{d}t'^2} \;\text{ in } S'.$$

Thus $\mathbf{F}' = \mathbf{F}$ and Newton's law is covariant wrt the Galilean transformation. Since the laws of conservation of energy and momentum follow from Newton's laws they are also Galilean covariant. Notice also that the spatial and temporal intervals between two events Δr and Δt are *separately* invariant under the Galilean transformation.

3 The Problem

Although Newtonian mechanics is a Galilean covariant theory, electromagnetism is *not*. For example the force between two charges at rest is not the same as the force between the same two charges in uniform motion. Also the Galilean transformation predicts that the velocity of light will depend on the velocity of the source, viz. $\mathbf{c}' = \mathbf{c} - \mathbf{v}$, and this is at variance with Maxwell's equations.

At the end of the 19th Century there were three possible solutions to the puzzle:

1. Galilean relativity correct. Newtonian mechanics correct. EM in need of modification.

2. Galilean relativity correct for mechanics. EM correct but in a preferred frame of reference at rest wrt the *aether*.

3. No preferred frames. Maxwell's equations correct and satisfying a principle of relativity (not Galilean). Newtonian mechanics in need of modification.

For a time the second proposal was the favourite but the failure to find any experimental evidence for the aether led to Einstein's theory of special relativity (1905).

Einstein based his theory on two fundamental postulates:

- The laws of physics are identical in all IFs — there are no preferred frames and there is no absolute motion.

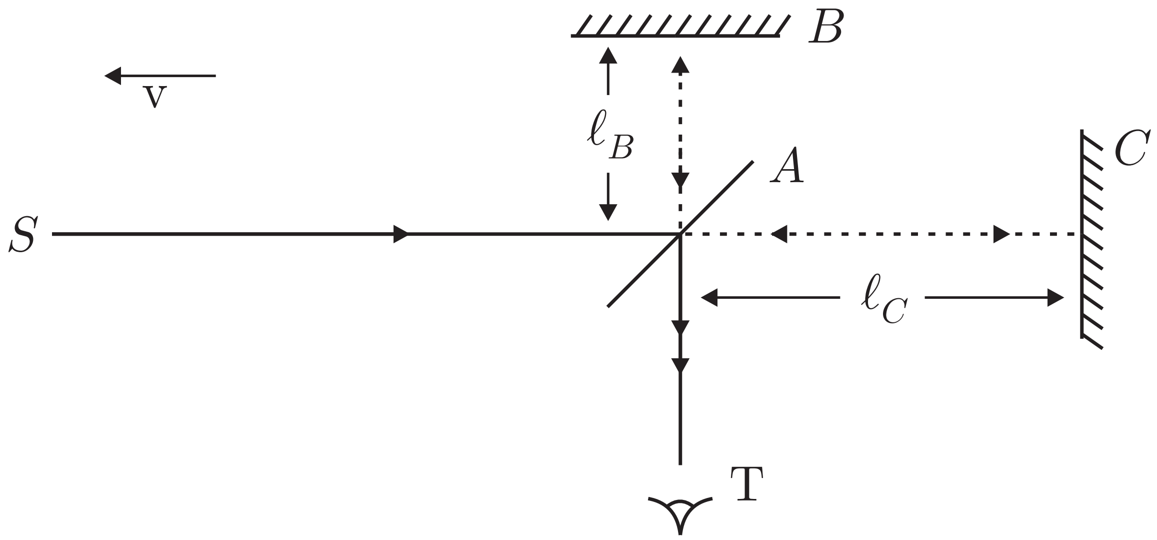
- The speed of light in a vacuum is a universal constant c with the same value in any IF — c does not change with the motion of source or observer.

All the physics of Special Relativity follows just from these two postulates.

4 Evidence for the constancy of c

a) Michelson-Morley Experiment:

High precision optical interference experiment to detect a change in the speed of light due to motion of the apparatus wrt the aether (as expected in Galilean relativity). The setup is that of a standard Michelson interferometer. Light from a source S is split into two beams by a half-silvered mirror at A. The beams are reflected from mirrors B and C and the final beam is viewed by telescope T.



Assume that the aether is moving wrt the apparatus with a velocity $\mathbf{v}$ parallel to AC as shown and that Galilean relativity is correct. Then:

$$\text{Time } A \rightarrow C \rightarrow A : \quad t_C = \frac{\ell_C}{c-v} + \frac{\ell_C}{c+v} = \frac{2\ell_C c}{(c^2-v^2)}.$$

During the time t it takes for light to traverse the distance AB, the mirror B will move a distance vt hence, $c^2t^2 = \ell_B^2 + v^2t^2$. The same is also true for the return path so:

$$\text{Time } A \rightarrow B \rightarrow A : \quad t_B = \frac{2\ell_B}{\sqrt{c^2-v^2}}.$$

Hence the time difference Δt is given by (for small v^2/c^2):

$$\Delta t = t_C - t_B = \frac{2(\ell_C - \ell_B)}{c} + \frac{v^2}{c^3}(2\ell_C - \ell_B). \qquad (I.1)$$

The whole apparatus is now rotated through 90^o so that $\mathbf{v}$ is now parallel to AB:

$$\Delta t' = t'_C - t'_B = \frac{2(\ell_C - \ell_B)}{c} + \frac{v^2}{c^3}(\ell_C - 2\ell_B).$$

Thus $\Delta t \neq \Delta t'$ and there should be a fringe shift δn in the interference pattern,

$$\delta n = \frac{c(\Delta t - \Delta t')}{\lambda} = \frac{v^2}{c^2}\frac{(\ell_C + \ell_B)}{\lambda}. \qquad (I.2)$$

In the original Michelson-Morley (1887) experiment $\ell_B = \ell_C = 11$ m and $\lambda = 600$ nm. No shift was observed, giving a limit of $\delta n < 0.01$ or $v < 5$ km/s to be compared, for example, with the Earth's orbital velocity of 30 km/s. The experiment has been repeated many times with null results of improving precision, some of which are shown in the Table.

Some Aether Drift Experiments:

Year	Experiment	Method	ℓ (m)	$v_{\max}$ (m/s)
1887	Michelson & Morley	Interferometer	11	5000
1903	Morley & Miller	Interferometer	32.2	380
1924	Miller (sunlight)	Interferometer	32	360
1924	Tomaschek (starlight)	Interferometer	8.6	2026
1927	Illingworth	Interferometer	2	173
1930	Joos	Interferometer	21	81
1963	Jaseja *et al*	He-Ne Maser	-	30
1963	Champeney *et al*	Mössbauer	-	5

On the basis of the electrodynamics of moving objects, Fitzgerald and Lorentz proposed that the length of an object moving parallel to the *aether wind* would be contracted by an amount $\sqrt{1 - v^2/c^2}$ so that $\ell_C \rightarrow \ell_C\sqrt{1 - v^2/c^2}$ and

$$\Delta t \;=\; t_C - t_B \;=\; \frac{2(\ell_C - \ell_B)}{c} \;+\; \frac{v^2}{c^3}(\ell_C - \ell_B). \qquad (I.3)$$

Thus for equal arms $\Delta t = 0$, similarly $\Delta t' = 0$, and no fringe shift would be expected.

This idea was disproved by the experiment of Kennedy & Thorndike (1932) in which an interferometer with *unequal* arms was used. Assuming the contraction hypothesis,

$$\Delta t \;\simeq\; \frac{2}{c}(\ell_C - \ell_B)(1 - v^2/c^2)^{-1/2}. \qquad (I.4)$$

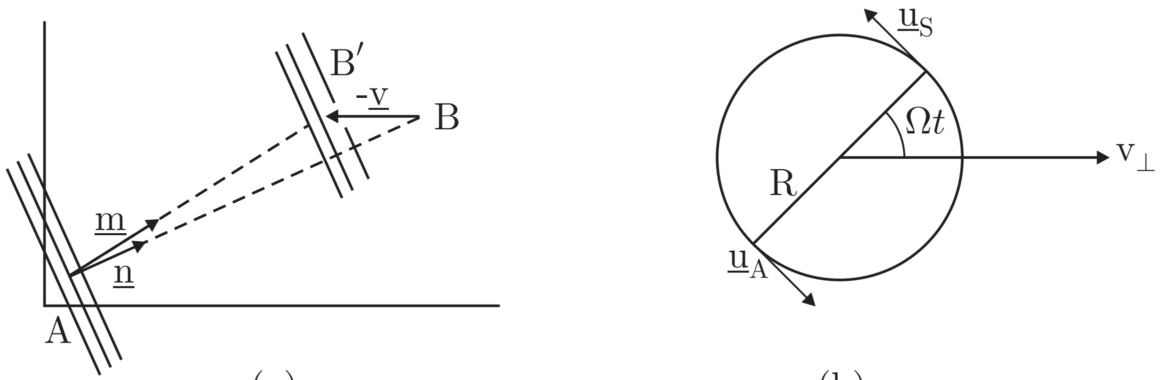
The apparatus itself was not rotated but a shift in fringe pattern would have been expected over a period of months as the Earth revolved around the Sun. None was however observed.

b) Mössbauer Experiment:

The Mössbauer effect is the *recoilless* emission and absorption of photons from atoms locked into a crystal lattice. The experiment of Champeney, Isaak & Khan (1963) used the photons from the 14.4 keV excited state of ^{57}Fe with a natural line width $\Delta\omega/\omega \sim 10^{-12}$. The idea of the experiment is to look for a frequency shift between a photon source and a resonant absorber placed at opposite ends of a high speed rotator. We discuss this experiment in detail since it provides the most sensitive test of aether drift to date.

The phase $\phi = \mathbf{k}.\mathbf{x} - \omega t$ of an EM wave is an invariant wrt to the Galilean transformation (it is proportional to the number of wave crests passing a fixed point in a fixed time). Considering our standard IFs S and S' we have:

$$\mathbf{k}'.\mathbf{x}' - \omega' t' \;=\; \mathbf{k}.\mathbf{x} - \omega t, \quad \text{or}$$



(a) (b)

$$\mathbf{k}'.(\mathbf{x}-\mathbf{v}t)-\omega' t \;=\; \mathbf{k}.\mathbf{x}-\omega t, \quad \text{hence}$$
$$\mathbf{k}' \;=\; \mathbf{k}, \quad \omega' \;=\; \omega(1-\frac{\mathbf{n}.\mathbf{v}}{c}). \tag{I.5}$$

The shift in phase velocity is $c' = c - \mathbf{n}.\mathbf{v}$, where $\mathbf{n}$ is the unit vector along $\mathbf{k}$. In addition the direction of the wave packet in the moving frame will be shifted to a direction $\mathbf{m}$ as seen from the fixed frame (see panel (a) in figure above),

$$\mathbf{m} \;=\; \frac{c\mathbf{n}-\mathbf{v}}{|c\mathbf{n}-\mathbf{v}|}.$$

To first order in v/c this can be inverted to give:

$$\mathbf{n} \;=\; \left(1-\frac{\mathbf{v}.\mathbf{m}}{c}\right)\mathbf{m} \;+\; \frac{\mathbf{v}}{c}. \tag{I.6}$$

If $\mathbf{u}_{S,A}$ are the lab velocities of the source and absorber respectively and $\mathbf{v}$ is the aether velocity the corresponding velocities relative to the aether are $\mathbf{v}_{S,A} = \mathbf{u}_{S,A} - \mathbf{v}$.

Using eq (I.5), we get

$$\omega_S - \omega_A \;=\; \frac{\omega}{c}(\mathbf{v}_A - \mathbf{v}_S).\mathbf{n} \;=\; \frac{\omega}{c}(\mathbf{u}_A - \mathbf{u}_S).\mathbf{n}.$$

Next refer $\mathbf{n}$ to the lab frame direction $\mathbf{m}$ to get,

$$\omega_S - \omega_A \;=\; \frac{\omega}{c}(\mathbf{u}_A - \mathbf{u}_S).(\mathbf{m}+\mathbf{v}/c). \tag{I.7}$$

In the experiment source and absorber were at the opposite ends of a rigid rotator (see panel (b) above) so $(\mathbf{u}_A - \mathbf{u}_S).\mathbf{m} = 0$ hence

$$\frac{\Delta\omega}{\omega} \;=\; \mathbf{v}.(\mathbf{u}_A - \mathbf{u}_S)/c^2 \;=\; \frac{2\Omega R}{c^2}\sin(\Omega t)|\mathbf{v}_\perp|, \tag{I.8}$$

using variables defined in panel (b) above and $|\mathbf{v}_\perp|$ as the component of the aether velocity perpendicular to the axis of rotation. Two rotation speeds of 1257 and 7728 rads/s were used with $R = 4$ cm. The observed fractional width was 2×10^{-12} and no diurnal change was seen to within $\sim 1\%$ giving $|\mathbf{v}_\perp| = 1.6 \pm 2.8$ m/s. This is clearly consistent with $v_\perp$, i.e. null evidence for an aether wind.

5 Independence of c from speed of the source

1) X-ray Binary Stars

The problem is to find sources moving at sufficiently high speeds to produce observable effects and for this reason binary star systems are, in principle, attractive. If the velocity of light depends on the velocity of the source (say as $c' = c + kv_{\rm source}$) then there will be deviations from the sinusoidal variation in the apparent velocity of the rotating star as measured by the Doppler effect. However interstellar space is not completely empty but contains a plasma with electron density $n_e \sim 0.05$ cm^{-3}, which will induce dispersive effects in a length $\chi = \lambda/2\pi|n-1|$, where n is the refractive index. In a plasma, this is $\chi = (\lambda\, n_{\rm e} a_0)^{-1}$ where $a_0 = e^2/(4\pi\epsilon_0 mc^2)$. Thus for an optical wavelength of 500 nm, $\chi \sim 2$ light years in interstellar space, i.e. less than the distance to even the nearest star. However for 70 keV X-rays, $\chi \sim 5 \times 10^4$ light years, the extent of the Galaxy. Using the X-ray binary Her X-1 at 1.9×10^4 light years, Brecher (1977) found a limit of $k < 2 \times 10^{-9}$. `(You may wish to attempt to repeat his analysis as described in Problem 4, Set 1.)`

2) $\pi^0 \;\rightarrow\; \gamma\gamma$ decay

The best terrestrial limit comes from the experiment of Alväger et al (1964) in which photons from π^0 decay were timed over an 80 m path length. The π^0 velocity was 0.99975c and k was found to be $(-3 \pm 13) \times 10^{-5}$, quite consistent with zero.

6 Independence of c from frequency

1) Crab Pulsar

The Crab pulsar (NP0532) is at a distance of $\sim 6 \times 10^3$ light years and has a pulse duration of $\sim 3 \times 10^{-3}$ s. Observations from frequencies of 4×10^8 Hz through the optical upto ~ 1 MeV photons gives $\Delta\, c/c \;<\; 10^{-14}$.

2) SLAC Experiment

Brown *et al* (1973) used the time of flight technique over a distance of 1300 m to compare the velocity of propagation of pulses of eV (optical) and 7 GeV photons and found $\Delta c/c \;=\; (1.8 \pm 6) \times 10^{-6}$. (They also compared the velocity of 11 GeV electrons with that of light, finding $(c - v_{\rm e})/c = (-1.3 \pm 2.7) \times 10^{-6}$, consistent with although not sensitive enough to test the special relativity expectation of $(c - v_{\rm e})/c \simeq m_{\rm e}^2/2E_{\rm e}^2 \sim 10^{-9}$.)

3) Supernova 1987A

Neutrinos of energy ~ 10 MeV were detected from the core collapse of a massive star 1.6×10^5 light years away in the Large Magellanic Cloud at 7:35:40 UT on 23 February 1987. Optical photons from the resulting supernova were first seen at 10:38 UT. (This delay is expected since the neutrinos are emitted in the first few seconds of the collapse while the outburst of light begins only after the stellar envelope is blown away after several hours by the outgoing shock wave.) Thus the fractional transit time difference of 2×10^{-9} gives an upper limit on the fractional difference in the velocity of propagation of light and neutrinos (which are very nearly massless).