# Topological Phases of Matter: Problem Set # 2

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## Problem 0 *About the Lowest Landau Level*

If you have never before actually solved the problem of an electron in two dimensions in a magnetic field, it is worth doing. Even if you have done it before, it is worth doing again. (If you have done it many times, it might not be worth doing yet again).

Consider a two dimensional plane with a perpendicular magnetic field $\vec{B}$. Work in symmetric gauge $\vec{A} = \frac{1}{2}\vec{r} \times \vec{B}$.

(a) (This is the hard part, see below for hints if you need them.) Show that the single electron Hamiltonian can be rewritten as

$$H = \hbar\omega_c(a^\dagger a \ + \frac{1}{2}) \tag{1}$$

where $\omega_c = eB/mc$ and

$$a = \sqrt{2}\ell \left(\bar{\partial} + \frac{1}{4\ell^2} z\right) \tag{2}$$

with $z = x + iy$ and $\bar{\partial} = \partial/\partial\bar{z}$ with the overbar meaning complex conjugation. Here $\ell$ is the magnetic length $\ell = \sqrt{\hbar c/eB}$.

(b) Confirm that

$$[a, a^\dagger] = 1 \tag{3}$$

and therefore that the energy spectrum is that of the harmonic oscillator

$$E_n = \hbar\omega_c(n + \frac{1}{2}) \tag{4}$$

(c) Once you obtain Eq. 1, show that any wavefunction

$$\psi = f(z)e^{-|z|^2/4\ell^2} \tag{5}$$

with $f$ any analytic function is an eigenstate with energy $E_0 = \frac{1}{2}\hbar\omega_c$. Show that an orthogonal basis of wavefunctions in the lowest Landau level (i.e., with eigenenergy $E_0$) is given by

$$\psi_m = N_m z^m e^{-|z|^2/4\ell^2} \tag{6}$$

where $N_m$ is a normalization constant. Show that the maximum amplitude of the wavefunction $\psi_m$ is a ring of radius $|z| = \ell\sqrt{2m}$ and calculate roughly how the amplitude of the wavefunction decays as the radius is changed away from this value.

(d) Defining further

$$b = \sqrt{2}\ell \left( \partial + \frac{1}{4\ell^2}\bar{z} \right) \tag{7}$$

with $\partial = \partial/\partial z$, Show that the operator $b$ also has canonical commutations

$$[b, b^\dagger] = 1 \tag{8}$$

but both $b$ and $b^\dagger$ commute with $a$ and $a^\dagger$. Conclude that applying $b$ or $b^\dagger$ to a wavefunction does not change the energy of the wavefunction, but applying $b^\dagger$ to a wavefunction generates a new wavefunction orthogonal to the original wavefunction.

(e) show that the $\hat{z}$ component of angular momentum (angular momentum perpendicular to the plane) is given by

$$L = \hat{z} \cdot (\vec{r} \times \vec{p}) = \hbar(b^\dagger b \;\; - a^\dagger a \;\; ) \tag{9}$$

Conclude that applying $b$ or $b^\dagger$ to a wavefunction changes its angular momentum, but not its energy.

## 0.1 *Hints to part a*

First, define the antisymmetric tensor $\epsilon_{ij}$, so that the vector potential may be written as $A_i = \frac{1}{2}B\epsilon_{ij}r_j$. We have variables $p_i$ and $r_i$ that have canonical commutations (four scalar variables total). It is useful to work with a new basis of variables. Consider the coordinates

$$\begin{aligned} \pi_i^{(\alpha)} &= p_i + \alpha\frac{\hbar}{2\ell^2}\epsilon_{ij}r_i & (10) \\ &= \frac{\hbar}{\ell^2}\epsilon_{ij}\xi_j & (11) \end{aligned}$$

defined for $\alpha = \pm 1$. Here $\alpha = +1$ gives the canonical momentum. Show that

$$\left[\pi_i^{(\alpha)}, \pi_j^{(\beta)}\right] = i\alpha\epsilon_{ij}\delta_{\alpha\beta}\frac{\hbar^2}{\ell^2} \tag{12}$$

The Hamiltonian

$$H = \frac{1}{2m}(p_i + \frac{e}{c}A_i)(p_i + \frac{e}{c}A_i) \tag{13}$$

can then be rewritten as

$$H = \frac{1}{2m}\pi_i^{(+1)}\pi_i^{(+1)} \tag{14}$$

with a sum on $i = \hat{x}, \hat{y}$ implied. Finally use

$$\begin{aligned} a &= (-\pi_y^{(+1)} + i\pi_x^{(+1)})\frac{\ell}{\sqrt{2}\hbar} & (15) \\ b &= (\pi_y^{(-1)} + i\pi_x^{(-1)})\frac{\ell}{\sqrt{2}\hbar} & (16) \end{aligned}$$

to confirm that $a$ and $b$ are given by Eqs. 2 and 7 respectively. Finally confirm Eq. 1 by rewriting Eq. 14 using Eqs. 15 and 16.

A typical Place to get confused is the definition of $\partial$. Note that

$$\partial z = \bar{\partial}\bar{z} = 1 \quad (17)$$
$$\bar{\partial} z = \partial\bar{z} = 0 \quad (18)$$

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## Problem 1 Quantum Hall Conductivity vs Conductance

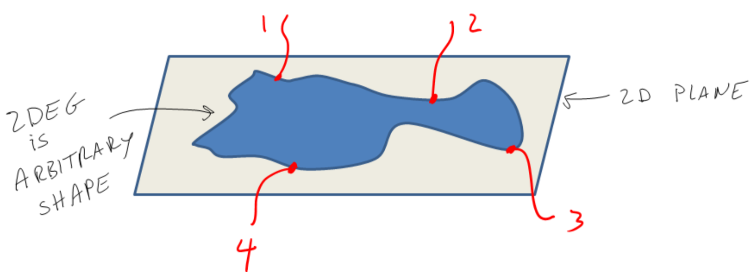


Figure 1: A 2DEG of arbitrary shape with contacts 1,2,3,4 attached on its perimeter in clockwise order

Consider a two dimensional electron gas (2DEG) of arbitrary shape in the plane with four contacts (1,2,3,4) attached at its perimeter in a clockwise order as shown in Fig. 1. The conductivity tensor $\sigma_{ij}$ relates the electric field to the current via

$$j_i = \sigma_{ij} E_j \quad (19)$$

where indices $i$ and $j$ take values $\hat{x}$ and $\hat{y}$ (and sum over $j$ is implied). Assume that this is a quantized hall system with quantized hall conductance $s$. In other words, assume that

$$\sigma = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix} \quad (20)$$

Show that the following two statements are true independent of the shape of the sample.

(a) Suppose current $I$ is run from contact 1 to contact 2, show that the voltage measured between contact 3 and 4 is zero.

(b) Suppose current $I$ is run from contact 1 to contact 3, show that the voltage measured between contact 2 and 4 is $V = I/s$.

Note: The physical measurements proposed here measure the *conductance* of the sample, the microscopic quantity $\sigma$ is the *conductivity*.

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## Problem 2 *Laughlin Plasma Analogy*

Consider the Laughlin wavefunction for $N$ electrons at positions $z_i$

$$\Psi_m^0 = \mathcal{N} \prod_{1 \le i < j \le N} (z_i - z_j)^m \prod_{1 \le i \le N} e^{-|z|^2/4\ell^2} \tag{21}$$

with $\mathcal{N}$ a normalization constant. The probability of finding particles at positions $\{z_1, \ldots, z_N\}$ is given by $|\Psi_m(z_1, \ldots z_N)|^2$.

Consider now $N$ classical particles at temperature $\beta = \frac{1}{k_b T}$ in a plane interacting with logarithmic interactions $v(\vec{r}_i - \vec{r}_j)$ such that

$$\beta v(\vec{r}_i - \vec{r}_j) = -2m \log(|\vec{r}_i - \vec{r}_j|) \tag{22}$$

in the presence of a background potential $u$ such that

$$\beta u(|\vec{r}|) = |\vec{r}|^2/(2\ell^2) \tag{23}$$

Note that this log interaction is "Coulombic" in 2d (i.e., $\nabla^2 v(\vec{r}) \propto \delta(\vec{r})$).

(a) Show that the probability that these classical particles will take positions $\{\vec{r}_1, \ldots, \vec{r}_N\}$ is given by $|\Psi_m^0(z_1, \ldots z_N)|^2$ where $z_j = x_j + iy_j$ is the complex representation of position $\vec{r}_i$. Argue that the mean particle density is constant up to a radius of roughly $\ell\sqrt{Nm}$. (Hint: Note that $u$ is a neutralizing background. What configuration of charge would fully screen this background?)

(b) Now consider the same Laughlin wavefunction, but now with $M$ quasiholes inserted at positions $w_1, \ldots, w_M$.

$$\Psi_m = \mathcal{N}(w_1, \ldots, w_M) \left[ \prod_{1 \le i \le N} \prod_{1 \le \alpha \le M} (z_i - w_\alpha) \right] \Psi_m^0 \tag{24}$$

where $\mathcal{N}$ is a normalization constant which may now depend on the positions of the quasiholes. Using the plasma analogy, show that the $w - z$ factor may be obtained by adding additional logarithmically interacting charges at positions $w_i$,with $1/m$ of the charge of each of the $z$ particles

(c) Note that in this wavefunction the $z$'s are physical parameters (and the wavefunction must be single-valued in $z$'s), but the $w$'s are just parameters of the wavefunction – and so the function $\mathcal{N}$ could be arbitrary — and is only fixed by normalization. Argue using the plasma analogy that in order for the wavefunction to remain normalized (with respect to integration over the z's) as the $w$'s are varied, we must have

$$|\mathcal{N}(w_1, \ldots, w_M)| = \mathcal{K} \prod_{1 \le \alpha < \gamma \le M} |w_\alpha - w_\gamma|^{1/m} \prod_{1 \le \alpha \le M} e^{-|w_\alpha|^2/(4m\ell^2)} \tag{25}$$

with $\mathcal{K}$ a constant. (Hint: a plasma will screen a charge).

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