Topological Phases of Matter: Problem Set # 1

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Problem 0 *About the Braid Group*

For several problems below I refer to the braid group. To define the braid group B_M for M particles, line up the M strands from left to right. The generator σ_m for $m = 1\ldots, M-1$ is a counter-clockwise exchange of particles m and $m+1$ as shown in Fig. 1 The braid generators may be composed in products and inverted as shown in

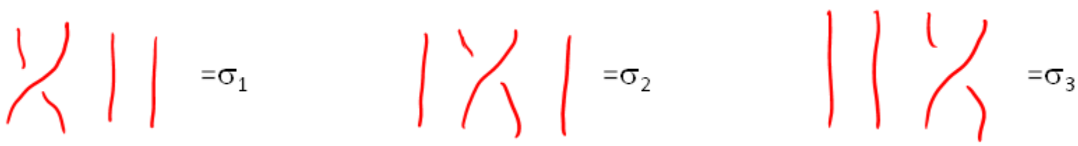

Figure 1: The three generators of the braid group on four strands, B_4

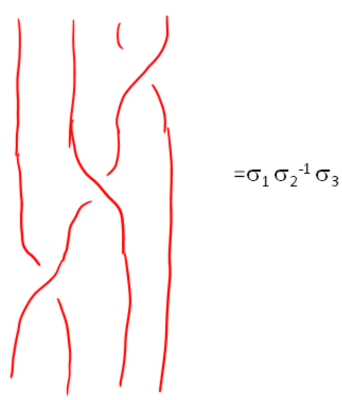

Figure 2: The three generators of the braid group on four strands, B_4

Fig. 2. Each braid (arbitrary product of the braid generators and their inverses) can be thought of as representing a class of topologically equivalent paths of M particles in a plane moving through time.

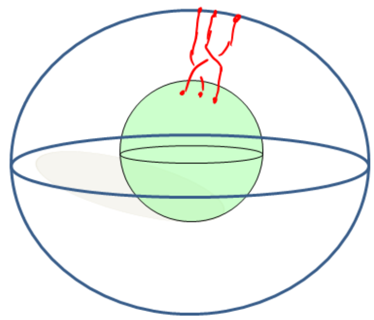

Figure 3: An element of the braid group $B_3(S^2)$. The braid shown here is $\sigma_1\sigma_2^{-1}$

(a) Convince yourself geometrically that the defining relations of the braid group are:

$$\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1} \qquad 1 \le i \le M-2 \tag{1}$$

$$\sigma_i\,\sigma_j = \sigma_j\,\sigma_i \quad \text{for} \quad |i-j|>1, \qquad 1 \le i,j \le M-1 \tag{2}$$

(b) Instead of thinking about particles on a plane, let us think about particles on the surface of a sphere. In this case, the braid group of M strands on the sphere is written as $B_M(S^2)$. To think about braids on a sphere, it is useful to think of time as being the radial direction of the sphere, so that braids are drawn as in Fig. 3. The braid generators on the sphere still obey Eqns. 1 and 2, but they also obey one additional identity

$$\sigma_1\sigma_2\ldots\sigma_{M-2}\sigma_{M-1}\sigma_{M-1}\sigma_{M-2}\ldots\sigma_2\sigma_1 = I \tag{3}$$

where I is the identity (or trivial) braid. What does this additional identity mean geometrically?

[In fact, for understanding the properties of anyons on a sphere, Eq. 3 is not quite enough. We will try to figure out below why this is using Ising Anyons as an example.]

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Problem 1 *Ising Anyons and Majorana Fermions*

The most commonly discussed type of non-Abelian anyon is the Ising anyon. This occurs in the Moore-Read quantum Hall state, as well as (potentially) in any chiral p-wave superconductor. The non-Abelian statistics of these anyons may be described in terms of Majorana fermions. [Majorana Fermions γ_j satisfy the anticommutation relation

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \tag{4}$$

as well as being self conjugate $\gamma_i^\dagger = \gamma_i$.] Ising anyons, can be represented by attaching a majorana to each anyon. The Hamiltonian for these majoranas is zero – they are completely noninteracting.

(a) Show that the ground state degeneracy of a system with $2N$ majoranas is 2^N if the Hamiltonian is zero. Thus conclude that each *pair* of Ising anyons is a two-state system.

(b) When anyon i is exchanged clockwise with anyon j, the unitary transformation that occurs on the ground state is

$$U_{ij} = \frac{e^{i\alpha}}{\sqrt{2}}\left[1 + \gamma_i \gamma_j\right] \qquad i < j. \tag{5}$$

for some real value of α. By confirming the braid relations Eq. 1 and 2, show that these unitary operators form a 2^N dimensional representation of the braid group.

(c) Consider the operator

$$\gamma^{\text{FIVE}} = (i)^N \gamma_1 \gamma_2 \ldots \gamma_{2N} \tag{6}$$

(the notation FIVE is in analogy with the γ^5 of the dirac gamma matrices). Show that the eigenvalues of γ^{FIVE} are ± 1. Further show that this eigenvalue remains unchanged under any braid operation. Conclude that we actually have two 2^{N-1} dimensional representations of the braid group. We will assume that any particular system of Ising anyons is in one of these two representations.

(d) Thus, 4 Ising anyons on a sphere comprise a single 2-state system, or a qubit. Show that by only braiding these four Ising anyons one cannot obtain any unitary operation on this qubit. Indeed, braiding Ising anyons is not sufficient to build a quantum computer. [Part d not required to solve part e,f]

(e) Now consider $2N$ Ising anyons on a sphere (See above problem about anyons on a sphere). Show that in order for either one of the 2^{N-1} dimensional representations of the braid group to satisfy the sphere relation, Eqn. 3, one must choose the right abelian phase α in Eq. 5. Determine this phase.

(f) The value you just determined is not quite right. It should look a bit unnatural as the abelian phase associated with a braid depends on the number of anyons in the system. Go back to Eqn. 3 and insert an additional abelian phase on the right hand side which will make the final result of part (e) independent of the number of anyons in the systm. In fact, there should be such an additional factor — to figure out where it comes from, go back and look again at the geometric "proof" of Eqn. 3. Note that the proof involves a self-twist of one of the anyon world lines. The additional phase you added is associated with one particle twisting around itself. The relation between self-rotation of a single particle and exchange of two particles is a generalized spin-statistics theorem.

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Problem 2 *Small Numbers of Anyons on a Sphere*

On the plane, the braid group of two particles is an infinite group (The free group with one generator). However, this is not true on a sphere

First review problem 0 about braiding on a sphere.

(a) Now consider the case of two particles on a sphere. Determine the full structure of the braid group. Show it is a well known finite discrete group. What group is it?

(b) Now consider three particles on a sphere. Determine the full structure of the braid group. Show that it is a finite discrete group. (Harder: What group is it? It is "well known" only to people who know a lot of group theory. But you can google to find information about it on the web with some work.)

Suppose we have two or three anyons on a sphere. Suppose the ground state is two-fold degenerate. If the braid group is discrete, conclude that no possible type of anyon statistics will allow us to do arbitrary $SU(2)$ rotations on this degenerate ground state by braiding.

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Problem 3 *One Approach to Exotic Statistics in 3+1 D*

As discussed in the class, the origin of exotic statistics is that the set of space-time paths separate into disconnected sets which cannot be smoothly deformed into each other. This allows us to associate different unitary operations with each topologically different space-time path.

In 3+1 D, for paths of point particles, the topologically different space-time paths are only the different permutations of which particle starts where and which ends where (this is just the statement that you cannot form a knot of one dimensional strings in four dimensions). However, if we imagine that our elementary particles are directed loops, rather than points then there are more possibilities. Let us suppose we have N such directed loops and we line them up and number them from left to right. Loops can be exchanged (as if they were point particles) or they can be threaded through each other (See figures 4 and 5). Let σ_i be a threading of $i+1$ through i

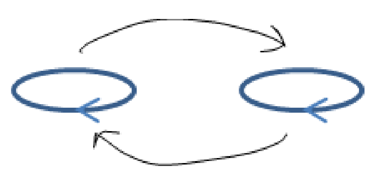

Figure 4: Exchanging loops as if they were simple point particles

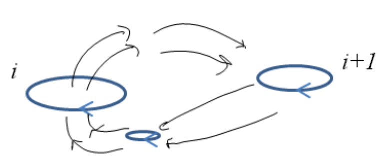

Figure 5: Threading loop $i+1$ through loop i

(Fig 5) followed by exchanging $i+1$ and i. Note that we have directed the loops (put arrows on them) so that a loop flipped over is not the same as the original loop.

Suppose, we have N directed-loop-like particle that are "bosonic" under simple exchange. (I.e., exchanging two such loops leaves the wavefunction unchanged). This particle may still have nontrivial "statistics" under threading as in Fig. 5. Show that the σ_i operators are isomorphic to the σ_i operators of the braid group B_N, that is, they obey Eqns. 1 and 2. Conclude that for any type of anyonic statistic that exists for point particles in 2+1 D, there can exist a corresponding "loop particle" statistics in 3+1 D.

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