SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 4 Year Course

B3: V. GENERAL RELATIVITY AND COSMOLOGY AND VI. CONDENSED-MATTER PHYSICS

TRINITY TERM 2011

Wednesday, 22 June, 2.30 pm – 5.30 pm

Answer **four** *questions,* **two** *from each section:*

Start the answer to each question in a **fresh book.**

At the end of the examination hand in your answers to Section V and Section VI in separate bundles.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section V. (General relativity and cosmology)

1. The space time metric around the Earth is

$$ds^2 = -c^2\alpha(r)dt^2 + [\alpha(r)]^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

with $\alpha(r) = 1 - \frac{2GM}{rc^2}$ and where M is the mass of the Earth, R is its radius and $\theta = 0$ corresponds to the North Pole. Geodesics in this space time satisfy:

$$\begin{aligned}
\frac{d}{d\lambda}\left(2c^2\alpha\dot{t}\right) &= 0 \\
\frac{d}{d\lambda}\left(2r^2\dot{\theta}\right) - 2r^2\sin\theta\cos\theta\dot{\phi}^2 &= 0 \\
\frac{d}{d\lambda}\left(2r^2\sin^2\theta\dot{\phi}\right) &= 0 \\
\frac{d}{d\lambda}\left(\frac{2\dot{r}}{\alpha}\right) - c^2\alpha'\dot{t}^2 + \frac{\alpha'}{\alpha^2}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2 &= 0
\end{aligned}$$

where λ is an affine parameter and $\alpha' = \frac{d\alpha}{dr}$. Consider Satellite A with a circular orbit, $\dot{r} = 0$, around the Earth along a line of constant $\dot{\phi} = 0$. Show that we are allowed to choose $\lambda = t$ and that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{GM}{r^3}\ .$$

[6]

Consider an observation station on the surface of the Earth at the North Pole. Find the proper time elapsed on Satellite A during an interval of time Δt as compared to the proper time elapsed on the observation station. Show that the ratio between the two can be approximated by

$$\frac{\Delta\tau_{\text{Satellite A}}}{\Delta\tau_{\text{North Pole}}} \simeq 1 + \frac{GM}{c^2R} - \frac{3}{2}\frac{GM}{c^2r}\ .$$

[10]

Now compare the proper time of a geostationary Satellite B on a circular orbit with $\theta = \pi/2$ with that on an observation station sitting on the equator right below it. Show that the ratio between the two proper times can be approximated by

$$\frac{\Delta\tau_{\text{Satellite B}}}{\Delta\tau_{\text{Equator}}} = 1 + \frac{GM}{c^2R} - \frac{GM}{c^2r}\ .$$

[5]

Explain the difference between the two results. [4]

2. Consider the metric

$$ds^2 = -c^2dt^2 + a^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \; .$$

Write down the Lagrangian for the geodesic equations in terms of an affine parameter λ. [3]

Show that the geodesic equations on this space time are

$$\begin{aligned}
\frac{d}{d\lambda}\left(-2c^2\dot{t}\right) &= +2a\frac{da}{dt}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) \\
\frac{d}{d\lambda}\left(2a^2\dot{r}\right) &= 2a^2r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \\
\frac{d}{d\lambda}\left(2a^2r^2\dot{\theta}\right) &= 2a^2r^2\sin\theta\cos\theta\dot{\phi}^2 \\
\frac{d}{d\lambda}\left(2a^2r^2\sin^2\theta\dot{\phi}\right) &= 0
\end{aligned}$$

where over-dots correspond to derivatives taken with respect to λ. Use these geodesic equations to write down all the non-zero connection coefficients for this space time. [10]

Now consider a geodesic along the radial direction and set $\theta = \pi/2$ and $\phi = 0$. Show that the geodesic equations can be solved to give

$$\begin{aligned}
\dot{r} &= \frac{\alpha}{a^2} \\
c^2\dot{t}^2 &= \frac{\alpha^2}{a^2} + \beta
\end{aligned}$$

where α and β are integration constants. Show that for a massless particle we have $\beta = 0$. [6]

Show that, if you choose $a(t) = \exp[H(t-t_0)]$ (where H is constant), the solution to the geodesic equations of a massless particle, emitted at time t_e from a distance r_e from the origin, is

$$\begin{aligned}
\exp[H(t-t_0)] &= \frac{H}{c}(\alpha\lambda + \epsilon) \\
r &= \frac{c^2}{H^2}\left[-\frac{1}{\alpha\lambda+\epsilon} + \delta\right] \; .
\end{aligned}$$

Combine these expression to show that at $t = t_0$

$$Ha(t_0)\, r_e = cz$$

with $1 + z = 1/R(t_e)$. [6]

3. The metric for a closed universe is given by

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-(r/R)^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

where the scale factor satisfies the FRW equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{c^2}{(Ra)^2} \; .$$

Here Λ is the cosmological constant and ρ is the energy density of dust. Draw a sketch showing how the three distinct contributions on the right hand side of the FRW equation evolve with time and the order in which each of them dominates. [4]

Consider $\Lambda = 0$ and assume that today, at $t = t_0$, we have that ρ is just very slightly larger than $\frac{3c^2}{8\pi G(Ra)^2}$. How will this Universe evolve in the future? Give a rough estimate, in terms of t_0, for how long you expect this Universe to last until a collapses to 0. [6]

Considering a radial geodesic for a photon, find the distance travelled as function of physical time, $r(t)$, as a function of the comoving distance,

$$D_c \equiv \int_0^t \frac{cdt'}{a(t')} \; .$$

What is the furthest distance a photon can travel in this universe? [6]

Now consider $\Lambda \neq 0$. Use the Raychauduri equation to find a static solution of the FRW equations. Show that in this Universe we have a particular value of the energy density, ρ_E, given by

$$\rho_E \equiv \frac{\Lambda c^2}{4\pi G} \; ,$$

and that the scale factor is given by

$$a = \sqrt{\frac{1}{R^2\Lambda}} \; .$$ [5]

Consider a universe for which ρ is just slightly larger than ρ_E. What is $\ddot{a}$ and how does this universe evolve? Consider now a universe for which ρ is just slightly smaller than ρ_E. What is $\ddot{a}$ and how does this universe evolve? Given what you found, do you think the static universe is stable? [4]

4. Consider a flat, homogenous and isotropic universe with scale factor $a(t)$ satisfying the FRW equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_{\mathrm{M}} + \rho_{\mathrm{R}}) \ ,$$

where ρ_{M} is the energy density in dust and ρ_{R} is the energy density in radiation. How do ρ_{M} and ρ_{R} depend on a ? Find a as a function of t at early times (when ρ_{R} dominates) and at late times (when ρ_{M} dominates) and ensure that that they match at equality, t_{eq}, when $\rho_{\mathrm{M}} = \rho_{\mathrm{R}}$. (Pick your constant of integration such that $a = 1$ today, at t_0.) [6]

Find an expression for the horizon size when $t < t_{\mathrm{eq}}$. Show that it takes the form

$$r_h(z) = \frac{c}{H_0}\frac{\sqrt{1+z_{\mathrm{eq}}}}{(1+z)^2} \ ,$$

where the redshift is defined through $1 + z = \frac{1}{a}$, z_{eq} is the redshift at equality and H_0 is the Hubble constant. [7]

The number density of photons in the Universe is given by

$$n_\gamma \ \simeq \ 0.486\,\frac{k_B T_0}{\hbar c}\frac{1}{a^3} \ \simeq \ \frac{8.3\times 10^8}{a^3}\,\mathrm{m}^{-3}$$

where $T_0 = 2.73\,\mathrm{K}$. Explain, using rough arguments, why this expression arises from the assumption of thermal equilibrium in the early Universe. What conditions in the early Universe ensure that there is thermal equilibrium? [5]

Find an expression for the total number of photons within one cosmological horizon, N_γ as a function of redshift for early times, assuming $z > z_{\mathrm{eq}}$ or $t < t_{\mathrm{eq}}$. Show that $N_\gamma \to 0$ as $z \to \infty$. Find an expression for the value of z when $N_\gamma = 1$ in terms of $N_\gamma(t_0)$. Explain why your result isn't compatible with the assumption of thermal equilibrium or with a homogeneous and isotropic Universe at very early times. [7]

Section VI. (Condensed Matter Physics)

5. Define the reciprocal lattice basis vectors $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ corresponding to a direct lattice with orthogonal basis vectors. [2]

The electron density ρ in a crystal has translational symmetry such that $\rho(\mathbf{r}) = \rho(\mathbf{r}+\mathbf{R})$ where $\mathbf{R}$ is a direct lattice vector: show that $\rho(\mathbf{r})$ can be written as

$$\rho(\mathbf{r}) = \sum_{\mathbf{G}} \rho_{\mathbf{G}} \exp\left(\mathrm{i}\mathbf{G}\cdot\mathbf{r}\right)$$

where $\mathbf{G} = (h\mathbf{b}_1 + k\mathbf{b}_2 + l\mathbf{b}_3)$ and h, k and l are integers. [6]

Derive an expression for the amplitude of X-ray scattering by a crystal in terms of the atomic form factors and the geometrical structure factor, and show that the change in wavevector of permitted X-ray reflections corresponds to a reciprocal lattice vector. [6]

GaAs has the cubic zincblende structure which has a face-centred cubic lattice with a basis of two atoms: Ga at $(0, 0, 0)$ and As at $(1/4, 1/4, 1/4)$ in the conventional cubic unit cell. Obtain an expression for the structure factor and find the conditions on $(h\,k\,l)$ for permitted X-ray reflections. [6]

Estimate the relative intensities of $(1\,1\,1)$ and $(2\,0\,0)$ reflections measured in a powder X-ray diffraction experiment. Give a physical explanation for the difference in intensities, and compare your result with that obtained for Si which has the same fcc lattice and atomic basis positions as GaAs. [The atomic numbers of Ga and As are: $\mathrm{Z_{Ga}} = 31$, $\mathrm{Z_{As}} = 33$.] [5]

6. A one-dimensional monatomic crystal contains identical atoms of mass m which experience a nearest-neighbour potential $V(x)$, where x is the distance between atoms. Under what conditions can $V(x)$ be approximated as a quadratic function of the displacement from the equilibrium position (harmonic approximation)? [4]

Analyse the dynamics of the crystal in terms of its normal modes of vibration in the harmonic approximation and obtain the dispersion relation between frequency and wavevector. Obtain an expression for the sound velocity v and indicate it on a sketch of the dispersion curve. [10]

The Lennard-Jones potential

$$V(x) = 4\varepsilon \left[\left(\frac{\sigma}{x} \right)^{12} - \left(\frac{\sigma}{x} \right)^{6} \right] ,$$

where x is the atomic separation and σ and ε are constants, provides a convenient description of the inter-atomic pair potential for rare gas solids. Assuming that only nearest-neighbour interactions are important, obtain expressions for the sound velocity and the maximum frequency of vibration in terms of the parameters ε and σ, and estimate their values in solid argon, given that $\varepsilon = 10\,\mathrm{meV}$ and $\sigma = 0.34\,\mathrm{nm}$. [11]

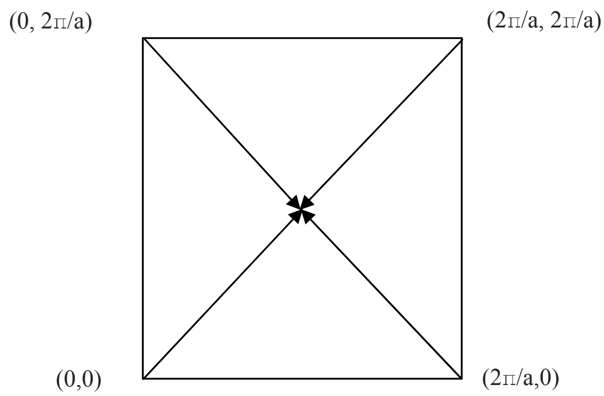
7. In the nearly free electron model electrons experience a weak periodic potential $V(\mathbf{r}) = V(\mathbf{r}+\mathbf{R})$, where $\mathbf{R}$ is any lattice vector. Show that $V_{\mathbf{k}-\mathbf{k}'} = \langle \mathbf{k}|V(r)|\mathbf{k}'\rangle$, where $|\mathbf{k}\rangle$ is the plane wave state, is non-zero only when $(\mathbf{k}-\mathbf{k}')$ is a reciprocal lattice vector $\mathbf{G}$. Using perturbation theory show that the secular equation is

$$(\varepsilon_0(\mathbf{k}) - E)\,(\varepsilon_0(\mathbf{k}+\mathbf{G}) - E) - |V_{\mathbf{G}}|^2 = 0\ ,$$ [9]

where ε_0 is the energy of $|\mathbf{k}\rangle$. Consider a divalent two-dimensional metal with a square lattice, lattice parameter $a = 0.3\,\mathrm{nm}$, and one atom per primitive unit cell. The periodic potential has two Fourier components V_{10}, V_{11}, corresponding to $\mathbf{G} = (1,0)$ and $(1,1)$ respectively; both are negative, and $|V_{10}| > |V_{11}|$.

(i) Write down the secular equation and obtain an expression for the electron energies at $\mathbf{k} = (\pi/a, 0)$. [4]

(ii) As illustrated in the diagram, the state at $(\pi/a, \pi/a)$ is four-fold degenerate in the free electron approximation, with energy ε_0.



In the presence of the periodic potential the secular equation is

$$\begin{vmatrix} \varepsilon_0 - E & V_{10} & V_{11} & V_{10} \\ V_{10} & \varepsilon_0 - E & V_{10} & V_{11} \\ V_{11} & V_{10} & \varepsilon_0 - E & V_{10} \\ V_{10} & V_{11} & V_{10} & \varepsilon_0 - E \end{vmatrix} = 0\ ,$$

which simplifies to

$$(\varepsilon_0 - E - V_{11})^2 \left[(\varepsilon_0 - E + V_{11})^2 - 4V_{10}^2\right] = 0\ .$$

Sketch the energy levels at $(\pi/a, \pi/a)$ and $(\pi/a, 0)$ in the same diagram. [4]

By considering overlapping energy bands, find the value of V_{10} at which the system becomes semiconducting given that $V_{11} = -0.2\,\mathrm{eV}$. [8]

8. The magnetization of a system of identical non-interacting magnetic ions, each with total angular momentum J in the presence of an applied magnetic field B at temperature T, is given by

$$M = ng\mu_B J\mathcal{B}_J\left(\beta g\mu_B JB\right)$$

where n is the number of ions per unit volume, g is the g-value, μ_B is the Bohr magneton, $\beta = 1/(k_\mathrm{B}T)$, and $\mathcal{B}_J(x)$ is the Brillouin function

$$\mathcal{B}_J(x) = \frac{2J+1}{2J}\coth\left[\frac{(2J+1)x}{2J}\right] - \frac{1}{2J}\coth\left[\frac{x}{2J}\right] .$$

(i) Sketch the function $\mathcal{B}_J(x)$ and explain the behaviour of M when $x \gg 1$. [3]

(ii) Show in the limit $x \ll 1$ that the susceptibility has the simple Curie form for a paramagnet, $\chi = C/T$, and obtain an expression for the Curie constant C. [5]

[You may use the approximation $\coth u \approx u^{-1} + u/3 + \mathrm{O}(u^3)$ for $u \ll 1$).]

Consider now that the ions interact with each other, which gives rise (in the mean field approximation) to an effective magnetic field $B_\mathrm{e} = \mu_0\lambda M$ that is proportional to the magnetisation, so that the total magnetic field $B_\mathrm{total} = B_\mathrm{applied} + B_\mathrm{e}$. Explain how this interaction gives rise to ferromagnetic behaviour and show that the susceptibility χ displays Curie-Weiss behaviour, $\chi = C/(T - T_\mathrm{C})$. Obtain an expression for the critical temperature T_C that marks the boundary between ferromagnetic and paramagnetic behaviour. [10]

Iron is a face-centred cubic ferromagnetic metal with $T_\mathrm{C} = 1043\,\mathrm{K}$, $J = 1$, $g = 2$ and $n = 8.5 \times 10^{28}\,\mathrm{m}^{-3}$. Estimate the values of λ and B_e. At an atomic level the effective field arises from an exchange interaction with neighbouring atoms. Assuming that the effective field arises entirely from nearest neighbour interactions, estimate the magnitude of the exchange interaction energy. [7]