Lecture notes: 3rd year fluids

Section H: Dynamical Systems

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Much of the material in these notes is based on Strogatz, *Non-linear dynamics and chaos*, which is an excellent introduction to the subject.

Contents

1 Introduction

The aim is to understand how systems, particularly non-linear systems, evolve in time. Consider the differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \ldots x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, \ldots x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \ldots x_n). \end{aligned} \tag{1}$$

Even if it is not possible to solve these explicitly, it is possible to learn useful things about the properties of the dynamical system by looking at flows or trajectories in the n-dimensional phase space $(x_1, x_2, \ldots x_n)$.

• Higher order DE's can be reduced to the form (1). For example

$$\ddot{x}_1 = -\sin x_1 \quad \text{is equivalent to} \quad \begin{aligned} &\dot{x}_1 = x_2 \\ &\dot{x}_2 = -\sin x_1. \end{aligned}$$

• DE's with time dependence can be reduced to the form (1). For example

$$\dot{x}_1 = -\sin x_1 + f(t) \quad \text{is equivalent to} \quad \begin{aligned} &\dot{x}_1 = -\sin x_1 + f(x_2) \\ &\dot{x}_2 = 1. \end{aligned}$$

• We shall also consider difference equations

$$x_{n+1} = f(x_n)$$

where n labels discrete time steps.

2 One-dimensional systems, $\dot{x} = f(x)$

2.1 Fixed points and linear stability analysis

At a fixed point x^* the system does not change with time, $\dot{x}(x^*) = f(x^*) = 0$.

Linearising about a fixed point gives its stability. Let

$$x = x^* + \eta$$

where x^* is the position of the fixed point and η is small. Then

$$\dot{x} = \dot{\eta} = f(x^* + \eta) = f(x^*) + \left.\frac{df}{dx}\right|_{x^*} \eta + \text{higher order terms.}$$

$f(x^*) = 0$ so

$$\dot{\eta} = \lambda \eta \quad \text{where} \quad \lambda = \left.\frac{df}{dx}\right|_{x^*}.$$

This means that

For $\lambda > 0$ any small perturbation away from the fixed point grows exponentially $\Leftrightarrow$ unstable fixed point.
For $\lambda = 0$ higher order terms are needed to draw any conclusion.
For $\lambda < 0$ any small perturbation away from the fixed point shrinks exponentially $\Leftrightarrow$ stable fixed point.

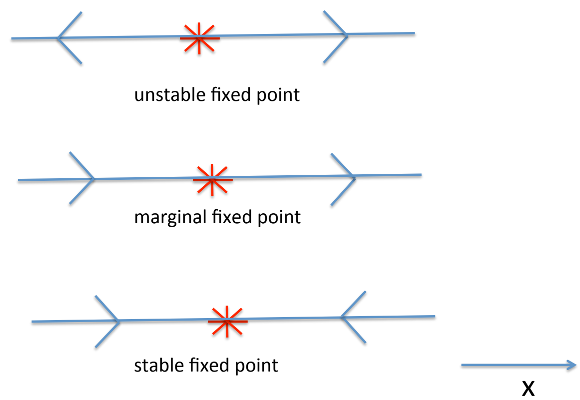


Figure 1: Fixed point stabilities.

2.2 Example: a simple population model

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) \tag{2}$$

where the first term represents exponential growth due to friendly animals and the second a limitation of resources that results from over-friendly animals. (Exhausted animals would just reduce r.) $r, K > 0$ are parameters describing the degree of friendliness /amount of food and $N > 0$ is the number in the population.

At fixed points $\dot{N} = 0$ so the fixed points are $N^* = 0$ and $N^* = K$.

Linearising around the fixed points to find their stability:

$$\frac{d\dot{N}}{dN} = r - \frac{2rN}{K} \Rightarrow$$

$$\left.\frac{d\dot{N}}{dN}\right|_0 = r > 0 \quad \therefore \text{unstable fixed point,}$$

$$\left.\frac{d\dot{N}}{dN}\right|_K = -r < 0 \quad \therefore \text{stable fixed point.}$$

We can use this information to sketch the flows in phase space. These show that, as expected, once a population is established it grows until it becomes resource limited.

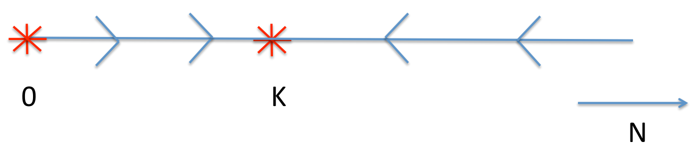


Figure 2: Flows in phase space for the population model (2).

2.3 Bifurcations

Bifurcations are qualitative changes in the behaviour of a dynamical system as a parameter is varied.

2.3.1 Saddle node bifurcation

This is the mechanism by which fixed points are created or destroyed. The canonical example is

$$\dot{x} = r + x^2. \tag{3}$$

Remember that x is the variable and r is a parameter: flows are in the phase space $\{x\}$, and they may change as the parameter r is varied.

To find the flows in phase space for the dynamical system (3) we find the fixed points, and their stabilities.

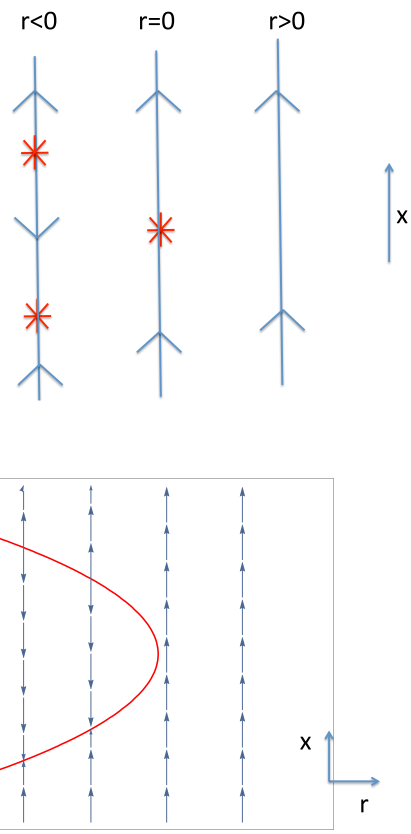


Figure 3: Top: flows in phase space for a saddle node bifurcation (3). Bottom: the flows are usefully plotted on a stability diagram which shows how they change as r is varied. It is usual to plot the stability diagram with x on the vertical axis and r on the horizontal axis. Note that the flows have to be vertical as r is a parameter that does not change in time, x is a variable that does.

fixed points

$$\dot{x} = 0 \qquad \Rightarrow \qquad x^2 = -r \qquad \Rightarrow$$

$$\begin{array}{lll} r > 0 & \text{no fixed points} & \\ r = 0 & \text{special case} & x^* = 0 \quad \text{(twice)} \\ r < 0 & \text{two fixed points} & x^*_+ = \sqrt{-r}, \quad x^*_- = -\sqrt{-r} \end{array}$$

check stability

$\frac{d\dot{x}}{dx} = 2x \quad \Rightarrow$

$$\left.\frac{d\dot{x}}{dx}\right|_{x^*_+} > 0 \;\; \therefore \text{unstable}, \qquad \left.\frac{d\dot{x}}{dx}\right|_{x^*_-} < 0 \;\; \therefore \text{stable}.$$

2.3.2 Transcritical bifurcation

Fixed points exchange stability at a transcritical bifurcation. The canonical form is

$$\dot{x} = rx - x^2.$$

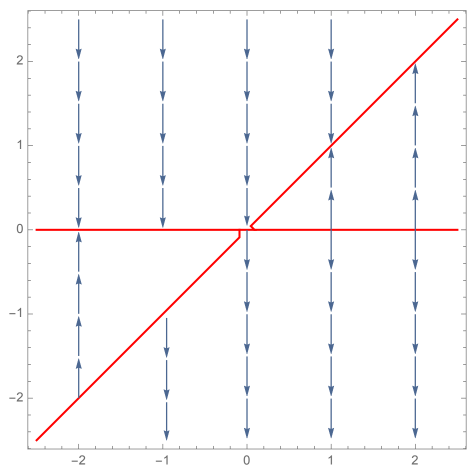


Figure 4: Stability diagram for a transcritical bifurcation.

2.3.3 Supercritical pitchfork bifurcation

A pitchfork bifurcation can occur if a system is unchanged when $x \Leftrightarrow -x$. The canonical form is

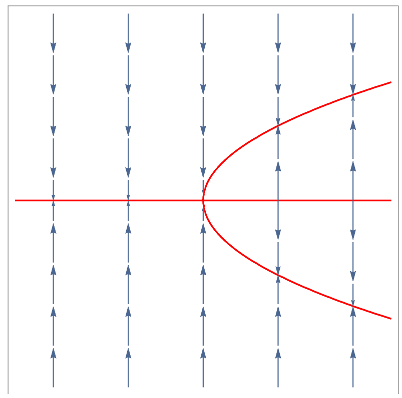
$$\dot{x} = rx - x^3. \tag{4}$$

For $r < 0$ there is a stable fixed point at $x^* = 0$. For $r > 0$ the fixed point $x^* = 0$ is unstable, and there are two stable fixed points at $x^* = \pm\sqrt{r}$. The stability diagram is shown in Figure 5.

The transition to Rayleigh-Bernard convection rolls is a supercritical pitchfork bifurcation. Near the transition

$$\tau \dot{A} = \epsilon A - gA^3 + \text{higher order terms that are small} \tag{5}$$

where τ is a time scale, A is the amplitude of eg the velocity and ϵ is a control parameter eg the temperature difference between the plates. It follows from Equation (5) that $A \sim \epsilon^{1/2}$ (Figure 5).



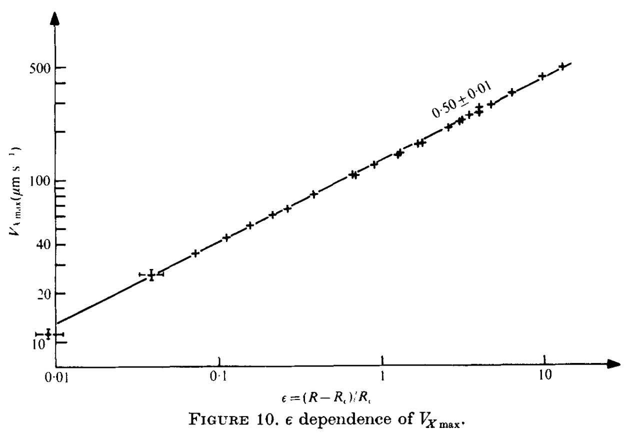


Figure 5: Top: stability diagram for a supercritical pitchfork bifurcation. Bottom: scaling near the onset of Rayleigh-Bernard convection

NB if the $x \Leftrightarrow -x$ symmetry is lost, the system undergoes an *imperfect pitchfork bifurcation* (see problem sheet).

2.3.4 Subcritical pitchfork bifurcation

The canonical form for a subcritical pitchfork bifurcation is

$$\dot{x} = rx + x^3. \tag{6}$$

The stability diagram is shown in Figure 6 (top).

For $r > 0$, x diverges to $\pm\infty$ with increasing time, so this is unlikely to be a good model of a physical system. A more physical example, that includes a subcritical pitchfork bifurcation and two saddle node bifurcations, is

$$\dot{x} = rx + x^3 - x^5. \tag{7}$$

The stability diagram, shown in Figure 6 (bottom), predicts hysteresis, a lack of reversibility as r is varied.

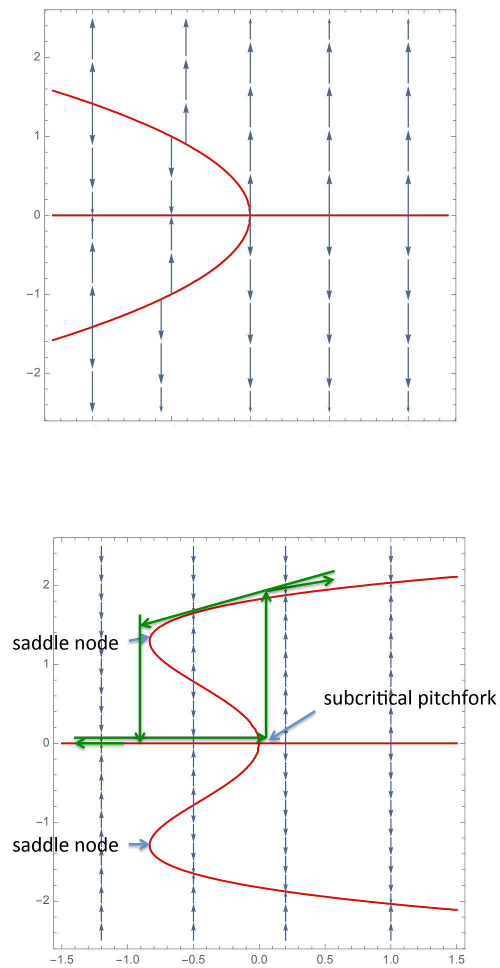


Figure 6: Stability diagrams for Equations (6) (top) and (7) (bottom) which contain subcritical pitchfork bifurcations.

3 Two-dimensional systems

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}$$

The phase plane is the space (x,y).
The phase portrait is the set of flows in the phase plane.

3.1 Fixed points and linear stability analysis

The fixed points are points where $\dot{x}=0$ and $\dot{y}=0$. We shall use linear stability analysis to investigate the flows near the fixed points.

Let

$$\begin{aligned}x &= x^*+\eta,\\ y &= y^*+\epsilon\end{aligned}$$

where η and ϵ are small. Then

$$(x^* \dot{+} \eta) = \dot{\eta} = f(x^*+\eta, y^*+\epsilon) = f(x^*,y^*) + \left.\frac{\partial f}{\partial x}\right|_* \eta + \left.\frac{\partial f}{\partial y}\right|_* \epsilon + \text{higher order terms},$$

$$(y^* \dot{+} \epsilon) = \dot{\epsilon} = g(x^*+\eta, y^*+\epsilon) = g(x^*,y^*) + \left.\frac{\partial g}{\partial x}\right|_* \eta + \left.\frac{\partial g}{\partial y}\right|_* \epsilon + \text{higher order terms}.$$

$f(x^*,y^*)$ and $g(x^*,y^*)$ are zero so, to leading order,

$$\begin{pmatrix}\dot{\eta}\\ \dot{\epsilon}\end{pmatrix} = \left.\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}\right|_* \begin{pmatrix}\eta\\ \epsilon\end{pmatrix} \equiv \mathcal{J}\begin{pmatrix}\eta\\ \epsilon\end{pmatrix}$$

where all the derivatives are evaluated at the fixed point and $\mathcal{J}$ is the Jacobian matrix.

Let $\mathcal{J}$ have eigenvalues λ_1, λ_2, and corresponding eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. If

$$\begin{pmatrix}\eta(0)\\ \epsilon(0)\end{pmatrix} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2$$

then it is easily checked that

$$\begin{pmatrix}\eta(t)\\ \epsilon(t)\end{pmatrix} = \alpha_1\mathbf{u}_1\exp\lambda_1 t + \alpha_2\mathbf{u}_2\exp\lambda_2 t.$$

3.2 Classification of fixed points

1. λ_1, λ_2 real and positive $\Leftrightarrow$ unstable node

2. λ_1, λ_2 real and negative $\Leftrightarrow$ stable node

3. $\lambda_1 > 0$, $\lambda_2 < 0$, both real $\Leftrightarrow$ saddle node

4. λ_1, λ_2 pure imaginary $\Leftrightarrow$ centre

5. λ_1, $\lambda_2 = \lambda_R \pm i\lambda_I$, $\lambda_R > 0 \Leftrightarrow$ unstable spiral

6. λ_1, $\lambda_2 = \lambda_R \pm i\lambda_I$, $\lambda_R < 0 \Leftrightarrow$ stable spiral

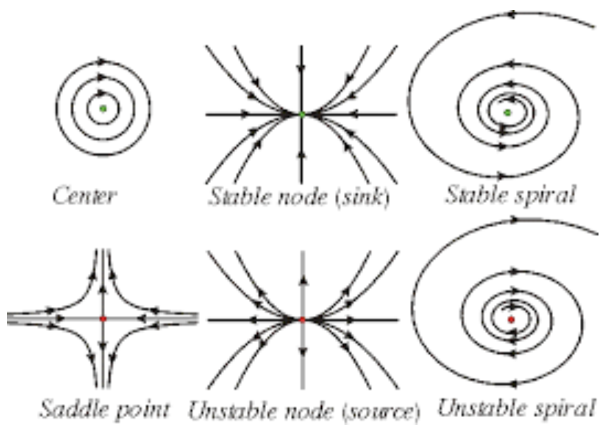


Figure 7: Fixed points in two dimensions.

3.3 To check that pure imaginary eigenvalues correspond to a centre

Consider the example

$$\dot{x} = -y,$$
$$\dot{y} = x.$$

This is simple harmonic motion so we expect trajectories $(\cos t, \sin t)$.

The fixed point is $(x^*, y^*) = (0, 0)$ and the Jacobian $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with eigenvalues and eigenvectors

$$\lambda_+ = i, \qquad \mathbf{u}_+ = (1, -i),$$
$$\lambda_- = -i, \qquad \mathbf{u}_- = (1, i).$$

For an initial perturbation

$$\begin{pmatrix} \eta(0) \\ \epsilon(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tfrac{1}{2}\mathbf{u}_+ + \tfrac{1}{2}\mathbf{u}_-$$

$$\begin{pmatrix} \eta(t) \\ \epsilon(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tfrac{1}{2}\mathbf{u}_+ e^{it} + \tfrac{1}{2}\mathbf{u}_- e^{-it}$$

$$= \tfrac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{it} + \tfrac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} e^{-it} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

3.4 Example 1: Lotka-Volterra preditor-prey model

$$\dot{R} = aR - bRF,$$
$$\dot{F} = -cF + dRF. \tag{8}$$

$a, b, c, d > 0$; $R, F > 0$.

Scaling variables, to minimise the number of free parameters, we take

$$R = \alpha x, \qquad F = \beta y, \qquad t' = \gamma t \quad \Rightarrow \quad \frac{d}{dt} = \gamma \frac{d}{dt'}.$$

Then the equations (8) become

$$\alpha\gamma x' = a\alpha x - b\alpha\beta xy, \tag{9}$$
$$\beta\gamma y' = -c\beta y + d\alpha\beta xy, \tag{10}$$

where $x' = \frac{dx}{dt'}$, etc. Reorganising equation (9) gives

$$x' = \frac{a}{\gamma}x - \frac{b\beta}{\gamma}xy,$$

so choose $\gamma = a$ and $\beta = a/b$ to give

$$\boxed{x' = x(1-y).}$$

Reorganising equation (10) gives

$$y' = -\frac{c}{\gamma}y + \frac{d\alpha}{\gamma}xy = -\frac{c}{a}y + \frac{d\alpha}{a}xy.$$

Choose $\alpha = c/d$ to give

$$\boxed{y' = \mu y(-1+x)}$$

where $\mu = c/a$.

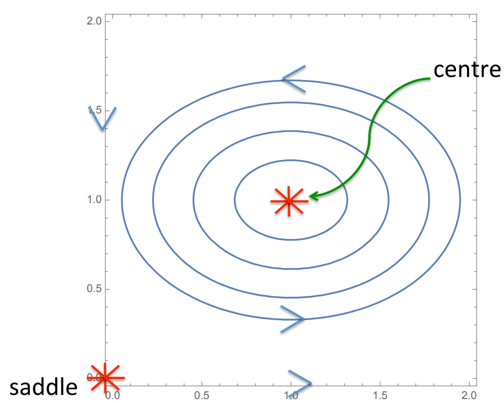


Figure 8: Phase portrait of the Lotka-Volterra model.

The fixed points are $(x^*, y^*) = (0,0)$ and $(1,1)$ and the Jacobian is

$$\mathcal{J} = \begin{pmatrix} 1-y & -x \\ \mu y & \mu(x-1) \end{pmatrix}.$$

If $x = y = 0$, $\mathcal{J} = \begin{pmatrix} 1 & 0 \\ 0 & -\mu \end{pmatrix}$ so $\lambda = 1, -\mu$ with eigenvectors $(1,0)$ and $(0,1)$ and this is a saddle.

If $x = y = 1$, $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ \mu & 0 \end{pmatrix}$ so $\lambda = \pm i\sqrt{\mu}$ and this is a centre.

- to find the directions of the flows test a few points.

- this model is unphysical because it doesn't pick out a preferred orbit.

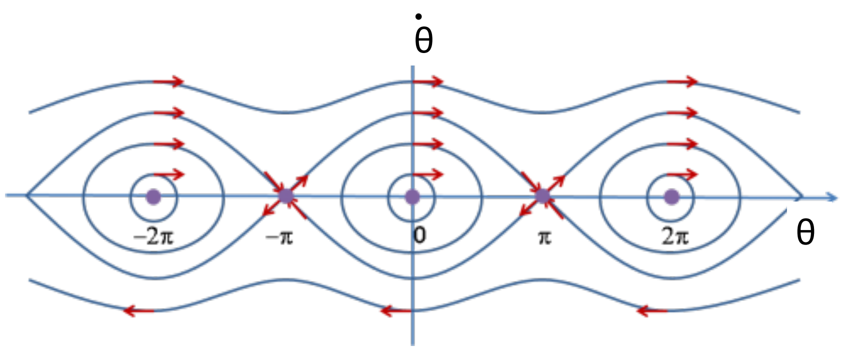


Figure 9: Phase portrait of a pendulum.

3.5 Example 2: The pendulum

$$\ddot{\theta} + \sin\theta = 0.$$

Rewrite as

$$\boxed{\dot{\theta} = \nu} \qquad \boxed{\dot{\nu} = -\sin\theta}$$

This has fixed points, $\nu = 0$, $\sin\theta = 0 \quad \Rightarrow \quad \theta = n\pi$.

The Jacobian is

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -\cos\theta & 0 \end{pmatrix} \text{ so } \lambda^2 = -\cos\theta \text{ and}$$

for $\theta = n\pi$, n even, $\lambda = \pm i$ corresponding to a centre.

for $\theta = n\pi$, n odd, $\lambda = \pm 1$, with eigenvectors $(1, 1)$ and $(1, -1)$, corresponding to a saddle.

The phase portrait is shown in Figure 9. Closed orbits correspond to oscillations around $\theta = 0$. Open orbits correspond to a circulating pendulum.

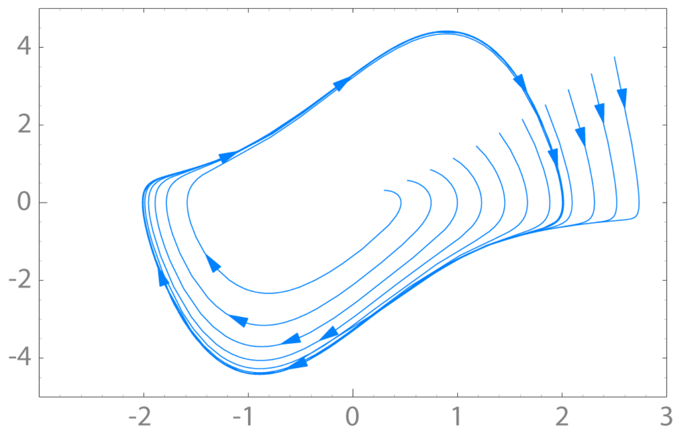


Figure 10: A stable limit cycle.

3.6 Limit cycles

A limit cycle is a closed trajectory in phase space which corresponds physically to an oscillation eg beating heart, firing of neurons. Figure 10 shows the flows around a stable limit cycle. The flows close to an unstable limit cycle away from the closed orbit.

• Limit cycles are inherently non-linear, which is why they did not appear in our classification of flows given in Sec. 3.2.
• They are not the same as a centre, which is not an isolated orbit.
• An example of a stable limit cycle is

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1.$$

3.7 Hopf bifurcations

The bifurcations we considered in Sec. 2.3 can be found in two- and higher-dimensional phase space: in a one-dimensional subspace with the new dimensions just giving simple attraction or repulsion. There are also new bifurcations which correspond to turning oscillations on or off.

• supercritical Hopf bifurcation

A stable spiral becomes an unstable spiral surrounded by a small limit cycle. Physically, oscillations that decay in time became oscillations that have constant amplitude in time.

• subcritical Hopf bifurcation

An unstable spiral shrinks and engulfs the origin which changes from a stable to an unstable fixed point. The trajectories can then fly off to a distant region of phase space. Physically, small decaying oscillations suddenly change their character. If the flows end on a large limit cycle they can be replaced by large amplitude oscillations. Turning the control parameter back down does not destroy the large oscillations because of hysteresis and this can cause problems in mechanical or biological systems.

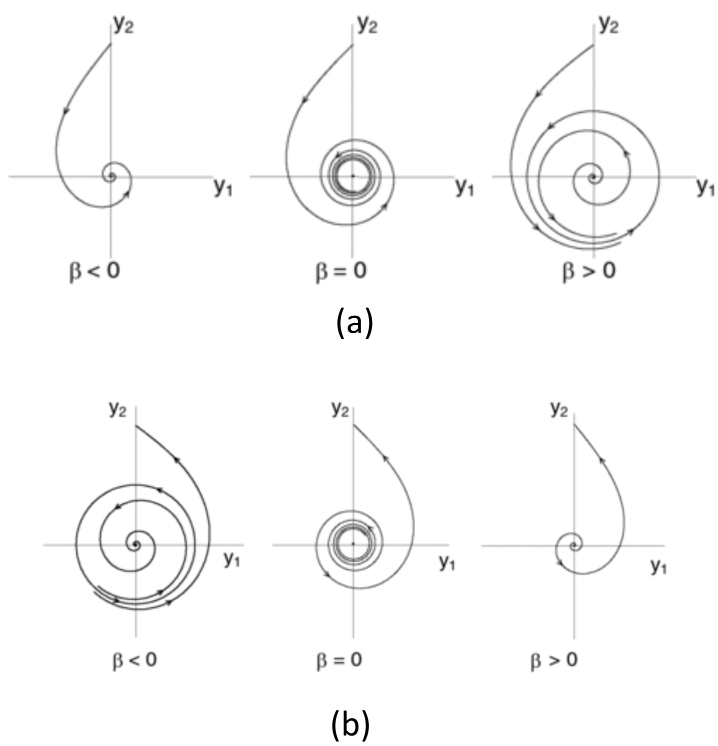


Figure 11: (a) Supercritical and (b) subcritical Hopf bifurcations.

3.8 Poincaré-Bendixson Theorem

In two dimensions (in phase space), if a trajectory is confined to a closed, bounded region that contains no fixed points then it must eventually approach a closed orbit.

But in three and higher dimensions trajectories may wander around forever in a closed region without reaching a fixed point or closed orbit. This allows chaos and strange attractors.

4 Higher dimensions and chaos

4.1 Background

4.1.1 Flows in phase space

How do volumes in phase space change with time? They can

• contract, eg a region in the vicinity of a stable fixed point
• expand, eg a region in the vicinity of an unstable fixed point
• stay the same, but change shape (cf incompressible flow in real space). This is the case for Hamiltonian systems.

Consider a closed volume $V(t)$. Points on the surface S move with velocity $\mathbf{u}(\mathbf{x}, t)$ and have normal $\hat{\mathbf{n}}(\mathbf{x}, t)$. Then

$$V(t+dt) = V(t) + \int_S \mathbf{u} \cdot \hat{\mathbf{n}}\, dS\, dt,$$
$$\frac{dV}{dt} = \frac{V(t+dt) - V(t)}{dt} = \int_S \mathbf{u} \cdot \hat{\mathbf{n}}\, dS = \int_V \nabla \cdot \mathbf{u}\, dV.$$

If $\nabla \cdot \mathbf{u}$ is constant

$$\frac{dV}{dt} = V\, \nabla \cdot \mathbf{u}. \qquad (11)$$

So, in a way analogous to incompressible flows, volumes in phase space are preserved if the divergence of the velocity is zero.

4.1.2 Liouville's theorem

A Hamiltonian system is characterised by a function $\mathcal{H}(x, y)$ where

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial y} \qquad\qquad \dot{y} = -\frac{\partial \mathcal{H}}{\partial x}. \tag{12}$$

Hamilton's equations (12) imply

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial x}\,\dot{x} + \frac{\partial \mathcal{H}}{\partial y}\,\dot{y} = 0.$$

For a Hamiltonian system

$$\frac{dV}{dt} = V\nabla \cdot \mathbf{u} = V\left(\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y}\right) = 0$$

showing that volumes are conserved in phase space.

4.1.3 Fractals

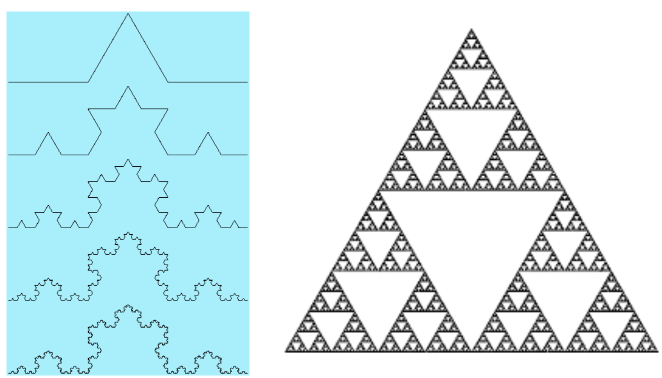


Figure 12: Rules for constructing regular fractals: (a) von Koch curve: replace the middle third of each line with the other two sides of an equilateral triangle; (b) Sierpinski carpet: remove central triangle.

Fractals are characterised by

- structure at all length scales
- self-similarity over a range of length scales

• a non-integer dimension

The similarity dimension is defined by

$$m = r^d \qquad \Rightarrow \qquad d = \frac{\ln m}{\ln r}$$

where m is the number of copies and r is the change in scale. So for the von Koch curve (Figure 12a) $m = 4$, $r = 3$ and $d = \ln 4/\ln 3 = 1.26$ and for the Sierpinski carpet (Figure 12b) $m = 3$, $r = 2$ and $d = \ln 3/\ln 2 = 1.59$.

Examples of fractals that are only approximately self-similar are: coastlines, mountain ranges, electrical discharges, broccoli, river deltas. To describe these a different way of defining the dimension is needed. To define the correlation dimension consider a fractal that is a set of points, and a ball (circle in 2D, sphere in 3D) of radius ϵ centred at $\mathbf{x}$. Then the number of points in the ball

$$n_{\mathbf{x}}(\epsilon) \propto \epsilon^{d(\mathbf{x})}.$$

In general d will depend on $\mathbf{x}$, but averaging over $\mathbf{x}$,

$$C(\epsilon) = \langle n_{\mathbf{x}}(\epsilon) \rangle_{\mathbf{x}} \propto \epsilon^d \tag{13}$$

where d is the correlation dimension. If the scaling (13) does not hold, then the set of points do not form a fractal. Usually d is found numerically by plotting $\ln C$ against $\ln \epsilon$. If there is a straight line over a sufficient range of ϵ (preferably several decades) its slope can be identified as the fractal dimension d. For small ϵ the scaling will break down because there are too few points in the test ball. For large ϵ the scaling will break down because the test ball is of a size comparable to that of the whole set.

4.2 Lorenz equations

4.2.1 Contraction of volumes in phase space

The Lorenz equations are a simplified model of convection rolls in the atmosphere:

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz. \end{aligned}$$

Parameters σ, related to the Prandtl number, r, related to the Rayleigh number, and b are positive.

This is a dissipative system, volumes in phase space contract exponentially fast.

$$\nabla \cdot \mathbf{u} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(\sigma + 1 + b).$$

Therefore, integrating Equation (11),

$$V(t) = V(0)e^{-(\sigma+1+b)t}$$

so at a long time the flows must approach

• a stable or saddle node,
or
• a stable or saddle-like limit cycle,
or
• something else with dimension in phase space that is < 3 – a strange attractor.

4.2.2 Fixed points and stability

The fixed points of the Lorenz equations are

• the origin $(0, 0, 0)$	which exists for all r. It is stable for $r < 1$.
• C^+: $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ • C^-: $(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$	which exist for $r > 1$. They are stable for $1 < r < r_H$ where $r_H = \frac{\sigma(\sigma+b+3)}{(\sigma-b-1)}$.

For $r < 1$ the origin is stable. At $r = 1$ there is a supercritical pitchfork bifurcation where the fixed point at the origin becomes unstable and a pair of symmetric stable fixed points are formed. At $r = r_H$ the stable fixed points undergo subcritical Hopf bifurcations (unstable cycle + stable fixed point $\Rightarrow$ unstable fixed point) and the trajectories fly to a distant ...

4.2.3 Strange attractor

Figure 13 shows the flows at $\sigma = 10$, $b = 8/3$, $r = 28$ (cf $r_H = 24.74$).

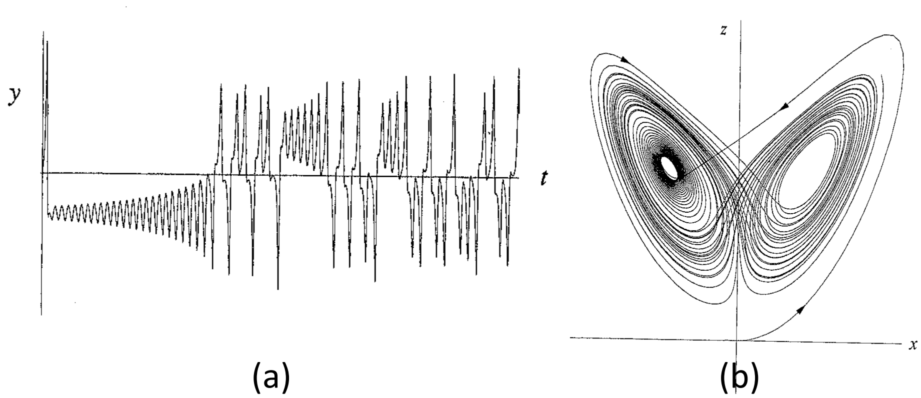


Figure 13: (a) $y(t)$ showing aperiodic, irregular oscillations that persist as $t \to \infty$ but never repeat. (b) Phase portrait projected onto (z, x). The oscillations switch unpredictably between the left-hand and right-hand loops. Note that this is a projection of a $>$ 2-dimensional structure onto 2 dimensions which is why it looks as if there are crossings. (c) The correlation dimension of the strange attractor is just above 2.

4.2.4 Exponential divergence of trajectories in phase space

We aim to find how the distance apart of two points in phase space $|\delta\mathbf{x}|$ evolves with time.

First note that

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{\overline{x + \delta x}} &= f(x + \delta x) \\ \therefore \dot{\delta x} &= f(x + \delta x) - f(x) = \frac{df}{dx}\delta x \end{aligned}$$

or, more generally, in higher dimensions

$$\frac{d(\delta\mathbf{x})}{dt} = \mathcal{J}(\delta\mathbf{x}). \tag{14}$$

Consider

$$\begin{aligned} \frac{d(|\delta\mathbf{x}|^2)}{dt} = \frac{d(\delta\mathbf{x}^T\delta\mathbf{x})}{dt} &= \delta\mathbf{x}^T \cdot \frac{d(\delta\mathbf{x})}{dt} + \frac{d(\delta\mathbf{x})^T}{dt} \cdot \delta\mathbf{x} \\ &= \delta\mathbf{x}^T(\mathcal{J} + \mathcal{J}^T)\delta\mathbf{x}. \end{aligned}$$

Assume $\delta\mathbf{x}$ is an eigenvector of $(\mathcal{J} + \mathcal{J}^T)/2$ with eigenvalue λ_i. Then

$$\frac{d(|\delta\mathbf{x}|^2)}{dt} = 2\lambda_i(\delta\mathbf{x})^T\delta\mathbf{x} = 2\lambda_i(|\delta\mathbf{x}|^2).$$

Integrating (assuming λ_i remains constant along the trajectory)

$$\boxed{|\delta\mathbf{x}| \sim e^{\lambda_i t}}.$$

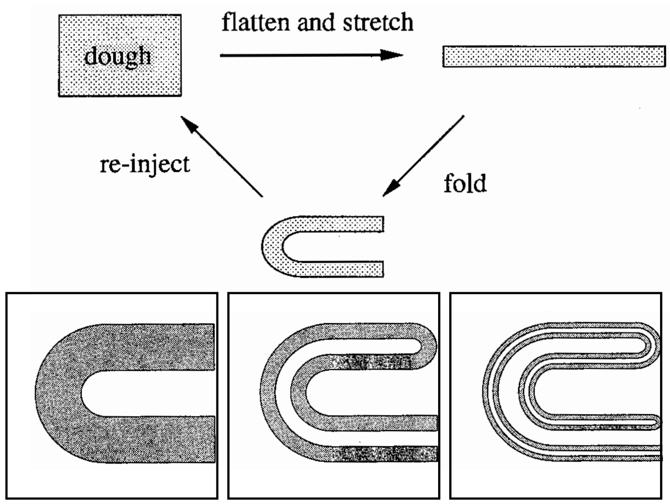


Figure: Strogatz (1994)

Figure 14: Stretching and folding of phase space leads to trajectories that initially separate exponentially fast but remain within a bounded region of phase space.

The λ_i are called the Liapunov exponents.

If any of the $\lambda_i > 0$, trajectories that start close together separate exponentially fast. Therefore it is not possible to predict the long-term behaviour of a chaotic system.

4.2.5 Chaos

Properties of a chaotic system:

• aperiodic long term behaviour (ie trajectories in phase space that do not settle down to fixed points or periodic orbits).

• sensitive dependence on initial conditions (at least one positive Liapunov exponent).

• deterministic (ie no noisy inputs, the irregular behaviour is a consequence of non-linear terms in the equations of motion).

Examples of chaotic systems:

population models, double pendulum, billiards on a on oval table, gravitational three-body problem, atmosphere

Mechanism for sensitive dependence on initial conditions:

Trajectories on a strange attractor remain confined to a bounded region of phase space, yet they initially separate exponentially fast. A mechanism for this is continual stretching and folding - like kneading dough (Figure 14).

Returning to the Lorenz equations:

For the parameters we are using $\lambda = -8/3$ corresponding to an eigenvector along z, and 10 (positive implying chaos), and -21, corresponding to eigenvectors in the (x, y) plane. Hence the dominant growth and decay is in the (x, y) plane.

4.3 Logistic map

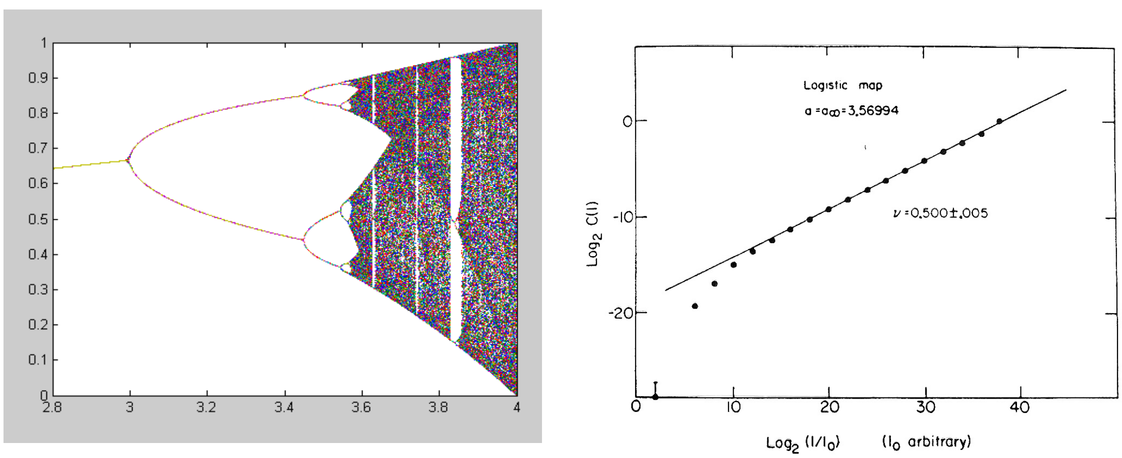


Figure 15: (a) Orbit diagram of the logistic map (b) Correlation dimension at r_∞ (from Grassberger and Procaccia (1983)).

The logistic map is a difference equations

$$x_{n+1} = r x_n (1 - x_n).$$

Consider $0 \leq r \leq 4$ so that the map remains in the interval $0 \leq x \leq 1$.

$r < 1$	$x_n^* = 0$ is a stable fixed point
$1 < r < 3$	$x_n^* = 1 - \frac{1}{r}$ is a stable fixed point
$r_1 = 3$	period 2 cycle born
$r_2 =$3.449...	period 4 cycle born
$r_3 =$3.544...	period 8 cycle born
$r_4 =$3.564 ...	period 16 cycle born
$\vdots$	
$r_\infty = 3.569946 \ldots$	period ∞ cycle born
$r_\infty < r < 4$	chaos interspersed with periodic windows

This is called the period doubling route to chaos.

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

$\delta = 4.669\ldots$ is a universal number, which is the same for a wide range of 1D maps. It has been measured experimentally for Rayleigh-Bernard convection at the transition between the roll state and turbulence and in electronic circuits that have a transition to chaos.