CALCULUS PROBLEMS

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Michaelmas Term

The problems are in 5 sections. The first 4, A Differentiation, B Integration, C Series and limits, and D Partial differentiation follow the lectures closly and it is recommended that all undergraduates attempt these. Part E contains problems which are slightly less standard or not on the syllabus. Tutors might like to set these for discussion in tutorials, or undergraduates who find the earlier problems straightforward might enjoy them.
= not on syllabus

0 Hyperbolic functions (for those who haven't met them before)

(a) Sketch $y = \sinh x, y = \cosh x$ and $y = \tanh x$ against x.

(b) Verify the following identities:

$$\cosh^2 x - \sinh^2 x = 1,$$

$$\cosh^2 x + \sinh^2 x = \cosh 2x, \qquad 2\cosh x \sinh x = \sinh 2x,$$

$$1 - \tanh^2 x = \mathrm{sech}^2 x, \qquad \coth^2 x - 1 = \mathrm{cosech}^2 x.$$

(c) Compare your results for (b) with trigonometric identities.

(d) Show that

$$\frac{d}{dx}(\sinh x) = \cosh x, \quad \frac{d}{dx}(\cosh x) = \sinh x, \quad \frac{d}{dx}(\tanh x) = \mathrm{sech}^2 x.$$

A. DIFFERENTIATION

A1 Practice in differentiation

(a) chain rule

Differentiate (i) $y = \sin x\, e^{x^3}$, (ii) $y = e^{x^3 \sin x}$, (iii) $y = \ln\{\cosh(1/x)\}$.

(b) inverse functions

Differentiate (i) $y = \cos^{-1} x$, (ii) $y = \tanh^{-1}\{x/(1+x)\}$.

(c) powers and logs

Differentiate (i) $y = x^{\cos x}$, (ii) $y = \log_{10}(x^2)$.

(d) implicit differentiation

(i) Find $\frac{dy}{dx}$ when $ye^{y \ln x} = x^2 + y^2$.
(ii) A particle moves a distance x in time t where

$$t = ax^2 + bx + c$$

with a, b, c constants. Prove that the acceleration is proportional to the cube of the velocity.
(e) parametric differentiation

(i) If $y = \sinh\theta$ and $x = \cosh\theta$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
(ii) If $y = t^m + t^{-m}$ and $x = t + t^{-1}$ show that

$$(x^2-4)\left\{\frac{dy}{dx}\right\}^2 = m^2(y^2-4), \qquad (x^2-4)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2 y = 0.$$

A2 Differentiation from first principles

Given the definition of the derivative as

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left\{\frac{y(x+\delta x) - y(x)}{\delta x}\right\}$$

evaluate $d(x^2)/dx$. In the same way evaluate $d(\sin x)/dx$.

A3 Integration as the inverse of differentiation

Given the function $I(x) = \int_a^x f(x')dx'$ outline a graphical argument that $dI(x)/dx = f(x)$. (Hint: sketch $y = f(x')$ and indicate the areas corresponding to $I(x)$ and $I(x+\delta x)$.)

A4 Derivatives of inverse functions

(a) Explain why

$$\frac{dx}{dy} = \left\{\frac{dy}{dx}\right\}^{-1}.$$

(b) Given that y is a function of x show, by putting $\frac{dy}{dx} = p$, that

$$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2} \Big/ \left(\frac{dy}{dx}\right)^3.$$

A5 <u>Changing variables in differential equations</u>

(a) For the differential equation

$$x^2\frac{d^2y}{dx^2} + (4x+3x^2)\frac{dy}{dx} + (2+6x+2x^2)y = x$$

replace the dependent variable y by $z = yx^2$ to give

$$\frac{d^2z}{dx^2} + 3\frac{dz}{dx} + 2z = x.$$

(ii) For the differential equation

$$4x\frac{d^2y}{dx^2} + 2(1-\sqrt{x})\frac{dy}{dx} - 6y = e^{3\sqrt{x}}$$

replace the independent variable x by $t = \sqrt{x}$ to give

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = e^{3t}.$$

These are equations with constant coefficients that you will soon be able to solve.

A6* <u>Leibnitz theorem</u>

Find the 8th derivative of $x^2 \sin x$.

B. INTEGRATION

B1 <u>Practice in integration</u>

Integrate the following:

(a) <u>inspection</u>

$$(i)\int \frac{(x+a)\,dx}{(1+2ax+x^2)^{3/2}}, \quad (ii)\int_0^{\pi/2} \cos x\, e^{\sin x} dx, \quad (iii)\int_0^{\pi/2} \cos^3 x\, dx, \quad (iv)\int_{-2}^{2} |\,x\,|\, dx.$$

(b) <u>change of variable</u>

$$(i)\int \frac{dx}{(3+2x-x^2)^{1/2}} \text{ (complete square first)}, \quad (ii)\int_0^{\pi} \frac{d\theta}{5+3\cos\theta} = \pi/4 \text{ (use } t = \tan\tfrac{\theta}{2}).$$

(c) <u>partial fractions</u>

$$\int \frac{dx}{x(1+x^2)}.$$

(d) parts

$$(i) \int x \sin x \, dx, \quad (ii) \int \ln x \, dx \quad \text{(write as } \textstyle\int 1. \ln x \, dx\text{)}.$$

(e) reduction

Prove that

$$\int_0^\infty x^n e^{-x^2} dx = \frac{1}{2}(n-1) \int_0^\infty x^{n-2} e^{-x^2} dx, \qquad n \geq 2$$

and hence evaluate

$$\int_0^\infty x^5 e^{-x^2} dx.$$

B2 Properties of definite integrals

(a) Which of the following integrals is zero? Explain why by sketching the integrand.

$$(i) \int_{-\infty}^{\infty} x e^{-x^2} dx, \qquad (ii) \int_{-\pi}^{\pi} x \sin x \, dx, \qquad (iii) \int_{-\pi}^{\pi} x^2 \sin x \, dx.$$

(b) Prove that, if $f(x)$ is an odd function of x,

$$\int_{-a}^{a} f(x) dx = 0.$$

(c) If $\ln x$ is defined by $\int_1^x t^{-1} dt$ show that

$$\ln x + \ln y = \ln xy.$$

B3 Arc length and area and volume of revolution

(a) Find the arc length of the curve $y = \cosh x$ between $x = 0$ and $x = 1$.

(b) Find the arc length of the curve $x = \cos t, \;\; y = \sin t$ for $0 < t < \pi/2$.

(c) Find the surface area and volume of a sphere of radius R by treating it as obtained by rotating the curve $y = \sqrt{R^2 - x^2}$ about the x-axis.

For (b) and (c) do you get the answers you expect?

B4 Line integrals

Evaluate the following line integrals:

(a) $\int_c (x^2 + 2y) dx$ from (0,1) to (2,3) where C is the line $y = x + 1$.

(b) $\int_c xy \, dx$ from (0,4) to (4,0) where C is the circle $x^2 + y^2 = 16$

(c) $\int_c (y^2 dx + xy dy + zx dz)$ from A(0,0,0) to B(1,1,1) where (i) C is the straight line from A to B; (ii) C is the broken line from A to B connecting (0,0,0), (0,0,1), (0,1,1) and (1,1,1).

C SERIES AND LIMITS

C1 Series notation

(a) Find a_n and b_n for $\frac{1}{4} - \frac{1}{8} + \frac{1}{16} - .... = \sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} b_n$. Sum the series.

(b) Write out the first few terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(c) Squares and products of whole series can also occur, for example $(a_1 + a_2 + a_3 + ...)^2$ and $(a_1 + a_2 + a_3 + ...) \times (b_1 + b_2 + b_3 + ...)$. How would you write these in $\sum$ notation?

C2 Maclaurin and Taylor series

(a) Find by differentiation the expansion of each of the following functions in power series in x up to and including terms in x^3:

$$(i)\ e^x, \qquad (ii) \sqrt{(1+x)}, \qquad (iii) \tan^{-1} x.$$

(b) Obtain the value of $\sin 31^o$ by expanding $\sin x$ to four terms about the point $x = \pi/6$. How precise is your answer?

C3 Manipulation of series

(a) From the series for $\sin x$ and $\cos x$ show that

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$$

(b) Using the power series for e^y and $\ln(1+x)$ find the first four terms in the series for $\exp\{(\ln(1+x)\}$, and comment on the result.

C4 Integration of a power series

Write down the power series expansion for $x^{-1} \sin x$. Hence evaluate, to four significant figures, the integral

$$\int_0^1 x^{-1} \sin x \, dx$$

C5 Continuity and differentiability

Sketch the following functions. Are they (i) continuous, (ii) differentiable, throughout the domain $-1 \leq x \leq 1$?

(a) $f(x) = 0$ for $x \leq 0$, $\quad f(x) = x$ for $x > 0$,
(b) $f(x) = 0$ for $x \leq 0$, $\quad f(x) = x^2$ for $x > 0$,
(c) $f(x) = 0$ for $x \leq 0$, $\quad f(x) = \cos x$ for $x > 0$,
(d) $f(x) = \mid x \mid$.

C6 Limits

Use (a) power series expansions, (b) L'Hôpital's rule to evaluate the following limits as $x \to 0$:

$$(i) \frac{\sin x}{x}, \qquad (ii) \frac{1 - \cos^2 x}{x^2}, \qquad (iii) \frac{\sin x - x}{\exp(-x) - 1 + x}.$$

(c) Find the limits of these expressions as $x \to \infty$.

(d) Expand $\{\ln(1+x)\}^2$ in power of x as far as x^4. Hence determine:

(i) whether $\cos 2x + \{\ln(1+x)\}^2$ has a maximun, minimum or point of inflection at $x = 0$.

(ii) whether

$$\frac{\{\ln(1+x)\}^2}{x(1-\cos x)}$$

has a finite limit as $x \to 0$ and, if so, its value.

D. PARTIAL DIFFERENTIATION

D1 Surfaces

(a) Sketch (in 3-dimensions) and (b) draw a contour map of the surfaces

(i) $z = (4 - x^2 - y^2)^{1/2}$,

(ii) $z = 1 - 2(x^2 + y^2)$,

(iii) $z = xy$,

(iv) $z = x^2 - y^2$.

D2 Getting used to partial differentiation

(a) Find $\frac{\partial f}{\partial x}$ for

$$(i) f = (x^2 + y^2)^{1/2}, \qquad (ii) f = \tan^{-1}(y/x), \qquad (iii) f = y^x.$$

(b) Verify that $f_{xy} = f_{yx}$ for

$$(i) f = (x^2 + y^2)\sin(x + y), \qquad (ii) f = x^m y^n.$$

(c) The function $f(x, y)$ is such that $f_{xy} = 0$. Find the most general forms for f_x and f_y and deduce that f has the form $f(x, y) = F(x) + G(y)$ where the functions F and G are arbitrary.

(d) If $V = f(x - ct) + g(x + ct)$ where c is a constant prove that

$$V_{xx} - \frac{1}{c^2}V_{tt} = 0.$$

D3 Error estimates

The acceleration of gravity can be found from the length l and period T of a pendulum; the formula is $g = 4\pi^2 l/T^2$. Using the linear approximation, find the relative error in g (i.e. $\Delta g/g$) in the worst case if the relative error in l is 5 % and the relative error in T is 2 %.

D4 Total derivatives

(a) Find $\frac{du}{dt}$ in two ways given that $u = x^n y^n$ and $x = \cos at$, $\quad y = \sin at$, where a, n are constants.

(b) Find $\frac{du}{dx}$ in two ways given that $u = x^2 y + y^{-1}$ and $y = \ln x$.

D5 Chain rule

If $w = \exp\{-x^2 - y^2\}$, $\quad x = r\cos\theta$, $\quad y = r\sin\theta$, find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ in two ways.

D6 Exact differentials

(a) The perfect gas law $PV = RT$ may be regarded as defining any one of the quantities pressure P, volume V or temperature T of a perfect gas as a function of the other two. (R = constant) Verify explicitly that

$$\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V = -1,$$

$$\left(\frac{\partial P}{\partial V}\right)_T = 1/\left(\frac{\partial V}{\partial P}\right)_T .$$

(b) Show that this is true whatever the relation $f(P, V, T) = 0$ between P, V and T.

D7 Change of variable (from Prelims 1997)

A variable z may be expressed either as a function of (u, v) or of (x, y), where $u = x^2 + y^2$, $v = 2xy$.

(a) Find

$$\left(\frac{\partial z}{\partial x}\right)_y \text{ in terms of } \left(\frac{\partial z}{\partial u}\right)_v \text{ and } \left(\frac{\partial z}{\partial v}\right)_u .$$

(b) Find

$$\left(\frac{\partial z}{\partial u}\right)_v \text{ in terms of } \left(\frac{\partial z}{\partial x}\right)_y \text{ and } \left(\frac{\partial z}{\partial y}\right)_x .$$

(c) Express

$$\left(\frac{\partial z}{\partial u}\right)_v - \left(\frac{\partial z}{\partial v}\right)_u \text{ in terms of } \left(\frac{\partial z}{\partial x}\right)_y \text{ and } \left(\frac{\partial z}{\partial y}\right)_x .$$

(d) Verify your expression explicitly in the case $z = u + v$.

D8 Taylor series in 2 variables

Expand $f(x, y) = e^{xy}$ to three terms around the point $x = 2, y = 3$.

D9 Stationary points

Find the position and nature of the stationary points of the following functions and sketch rough contour graphs in each case.

$$(i) f(x, y) = x^2 + y^2, \qquad (ii) f(x, y) = x^3 + y^3 - 2(x^2 + y^2) + 3xy,$$

$$(iii) f(x, y) = \sin x \sin y \sin(x + y), \quad 0 < x < \pi/2; \quad 0 < y < \pi/2.$$

D10 Exact differentials

(a) Which of the following are exact differentials? For those that are exact, find f.

$$(i) df = xdy + ydx, \quad (ii) df = xdy - ydx, \quad (iii) df = xdx + ydy + zdz.$$

(b) What is the value of $\oint xdy + ydx$ around the curve $x^4 + y^4 = 1$?

E. ADDITIONAL PROBLEMS

E1 Binomial expansion

The relativistic expression for the energy of a particle of mass m is

$$E = \frac{mc^2}{(1 - v^2/c^2)^{1/2}}$$

where v is the particle velocity and c the speed of light. Expand this $O(v^4/c^4)$ and identify the terms you obtain.

E2 Newton's method

If x_i is an approximation to a root of the equation $f(x) = 0$, Newton's method of finding a better approximation x_{i+1} is $x_{i+1} = x_i - f(x_i)/f'(x_i)$, where $f'(x) = df/dx$. Explain this method graphically or otherwise in terms of the linear approximation to $f(x)$ near $x = x_i$.

E3 Evaluating derivatives numerically

Use Taylor's theorem to show that when h is small

(a) $f'(a) = \frac{f(a+h)-f(a-h)}{2h}$ with an error $O(h^2 f'''(a))$.

(b) $f''(a) = \frac{f(a+h)-2f(a)+f(a-h)}{h^2}$ with an error $O(h^2 f''''(a))$.

Taking $f(x) = \sin x$, $a = \pi/6$, and $h = \pi/180$ find from (a) and (b) the approximate values of $f'(a)$ and $f''(a)$ and compare them to exact values.

* These finite-difference formulae are often used to calculate derivatives numerically. How would you construct a more precise finite-difference approximation to $f'(a)$?

*E4 More on differentiability

Sketch the graph of

$$f(x) = e^{-x} + 2x, \qquad x \geq 0; \qquad f(x) = e^x, \qquad x < 0$$

and sketch its 1st, 2nd and 3rd derivatives. Show that the third derivative is discontinuous at $x = 0$.

E5 More limits

Find

$$(i) \lim_{x\to -1} \frac{\sin \pi x}{1+x}, \qquad (ii) \lim_{x\to\infty} \frac{2x\cos x}{1+x}, \qquad (iii) \lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x},$$

$$(iv) \lim_{x\to 0} \frac{\sec x - \cos x}{\sin x}, \qquad (v) \lim_{x\to\pi/2} (\sin x)^{\tan x}.$$

*E6 Leibnitz theorem and McClaurin series (from Prelims 1999)

For the function

$$y = \cos(a \cos^{-1} x)$$

show that

$$(1 - x^2)y'' - xy' + a^2 y = 0 \tag{1}$$

where a is a constant.

Use Leibnitz' theorem to differentiate (1) n times and then put $x = 0$ to show that for $n \geq 0$

$$y^{(n+2)}(0) = (n^2 - a^2)y^{(n)}(0)$$

where $y^{(n)}(0)$ is the n^{th} derivative of $y(x)$ evaluated at $x = 0$.

Use this result to obtain a terminating power series expansion for $y = \cos(3\cos^{-1} x)$ in terms of x. Verify that your solution solves (1).

E7* Change of variables

Spherical polar coordinates (r, θ, ϕ) are defined in terms of Cartesian coordinates (x, y, z) by

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta.$$

(a) Find $(\partial x/\partial r)$, treating x as a function of the spherical polar coordinates, and $(\partial r/\partial x)$ treating r as a function of the Cartesian coordinates.

(b) Given that f is a function of r only, independent of θ and ϕ, show that

$$\frac{\partial f}{\partial x} = \frac{x}{r}\frac{df}{dr},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{r}\frac{df}{dr} + \frac{x^2}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{df}{dr}\right),$$

and hence deduce that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df}{dr}\right).$$