Electromagnetic Waves

E. Plasmas

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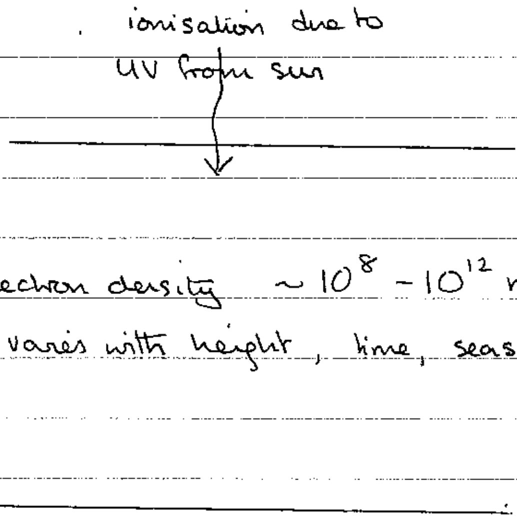
E. Plasmas

1. Introduction

A plasma is a gas of ionised particles which is electrically neutral.

e.g. free electrons + ions

a. ionosphere



free electron density $\sim 10^{8} - 10^{12}\ m^{-3}$

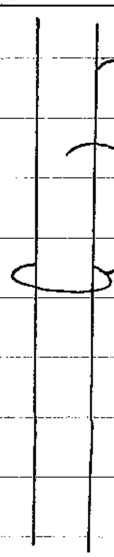
varies with height, time, season, sunspot cycle .....

b. stars & fusion reactors

light atoms compressed & heated to try to reproduce conditions in stars interiors; at these temps. they are ionised

c. space (very dilute plasma)

d. plasma displays (large TV displays)



two plates of glass

tiny cells contains inert gases

voltage across plates ionises gases

gas ions collide with electrodes + emit UV

UV hits phosphors on back of each cell

$\Rightarrow$ colours

2. some reminders

remember: (i) $n \propto \sqrt{\epsilon}$ (if $\mu = 1$)

because $$n = \frac{c}{v} = \frac{\sqrt{\epsilon_0 \mu_0 \epsilon \mu}}{\sqrt{\epsilon_0 \mu_0}} \sim \sqrt{\epsilon}$$

(we will generalise this for complex n, ϵ)

(ii) a wave propagates as

$$e^{j(\omega t - \tilde{k} z)}$$

$$= e^{j\left(\omega t - \frac{\tilde{n}\omega z}{c}\right)} \qquad \tilde{n} = n_R - j n_I$$

$$= e^{j\omega\left(t - \frac{n_R z}{c}\right)} \cdot e^{-\frac{n_I \omega z}{c}}$$

n_R controls speed of propagation

n_I controls adsorption

(iii) $$\underline{D} = \epsilon_0 \underline{E} + \underline{P} \equiv \epsilon \epsilon_0 \underline{E}$$

$$\therefore \underline{P} = (\epsilon - 1)\epsilon_0 \underline{E}$$

3. refractive index of a plasma: method 1

consider a free charge in an oscillating field
(only have to worry about the electrons moving; ions are much heavier and, to a good approx. stationary)

$$qE = m\ddot{x} \qquad (1)$$

if $E = E_0 e^{j\omega t}$

$$-\frac{jqE_0 e^{j\omega t}}{m\omega} = \dot{x}$$

$$-\frac{qE_0 e^{j\omega t}}{m\omega^2} = x$$

$\therefore$ conductivity $\sigma = \frac{Nq\dot{x}}{E} = -\frac{jNq^2}{m\omega}$ (2) (needed later)

polarisation $P = Nqx = -\frac{Nq^2 E}{m\omega^2} = (\epsilon - 1)\epsilon_0 E$

(3)

$$\therefore \epsilon = 1 - \frac{Nq^2}{\omega^2 m \epsilon_0}$$

$$\boxed{\tilde{n}^2 = 1 - \frac{Nq^2}{\omega^2 m \epsilon_0} \equiv 1 - \frac{\omega_p^2}{\omega^2}}$$

where $\omega_p^2 = \frac{Nq^2}{m\epsilon_0}$ is the plasma frequency

$\omega < \omega_p$ $\tilde{n}$ is purely imaginary + em waves cannot propagate
$\omega > \omega_p$ " real " can "

for ionosphere plasma freq typically ~ 1 MHz
AM radio refracted + reflected (~1 MHz)
FM + television refracted + escape (~100 MHz) varies with height as N changes

4. refractive index of a plasma: method 2

In method 1 we assumed the plasma was 'polarised' i.e acting as a dielectric. This is worrying as there is no bound charge — get away with it because the electrons are oscillating about fixed positions in this simple model.

Can get the same answer if we treat the plasma as having conductivity $\sigma = -\frac{jNq^2}{\omega m}$ (eq^n (2))

but the calculation is slightly more tricky.

σ is complex so we need to go back to the dispersion relation for conductors (from subst. plane wave soln into the wave eqn for conductors

$$\tilde{k}^2 = \mu_0 \epsilon_0 \omega^2 - j \mu_0 \sigma \omega$$

$$\therefore \tilde{k}^2 = \frac{\omega^2}{c^2}\left(1 - \frac{j\sigma}{\omega \epsilon_0}\right) \qquad \text{using} \quad c^2 = \frac{1}{\epsilon_0 \mu_0}$$

$$= \frac{\omega^2}{c^2}\left(1 - \frac{Nq^2}{\epsilon_0 \omega^2 m}\right)$$

$$\therefore \tilde{n}^2 = \left(\frac{c\tilde{k}}{\omega}\right)^2 = 1 - \frac{Nq^2}{\epsilon_0 \omega^2 m} \qquad \text{as before}$$

5. to check that methods 1, 2 are different ways of looking at the same thing:

$$\text{curl } \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t} \qquad \text{assume } E \sim E_0 e^{j\omega t}$$

$$= (\sigma + j\epsilon\epsilon_0\omega)\,\underline{E}$$

method 1: $\quad \sigma = 0 \qquad \epsilon = 1 - \frac{\omega_p^2}{\omega^2}$

$$\therefore \text{curl } \underline{H} = j\omega\epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)\underline{E}$$

method 2: $\quad \sigma = -\frac{j\omega_p^2 \epsilon_0}{\omega} \qquad \epsilon = 1$

$$\therefore \text{curl } \underline{H} = \left(-\frac{j\omega_p^2\epsilon_0}{\omega} + j\epsilon_0\omega\right)\underline{E}$$

$$= j\omega\epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)\underline{E}$$

ie this (+ the other) Maxwell's equations are identical

6. Example

Pulsar emits radio waves. Interval between times of arrival at earth of pulses at 400 MHz and 200 MHz is 4 s. electron density in interstellar space is $3 \times 10^4 \, m^{-3}$. What is the distance to the pulsar?

signals travel with group velocity

$$\omega^2 = \frac{c^2 k^2}{n^2} = \frac{c^2 k^2}{1 - \frac{\omega_p^2}{\omega^2}}$$

$$\therefore \omega^2 = \omega_p^2 + c^2 k^2$$

$$2\omega \frac{d\omega}{dk} = 2c^2 k$$

$$v_g = \frac{d\omega}{dk} = \frac{c^2 k}{\omega} = \underline{\frac{c\sqrt{\omega^2 - \omega_p^2}}{\omega}}$$

time difference between signals

distance to pulsar

$$\Delta t = \frac{d\,\omega_1}{c\sqrt{\omega_1^2 - \omega_p^2}} - \frac{d\,\omega_2}{c\sqrt{\omega_2^2 - \omega_p^2}}$$

working out $f_p = \frac{\omega_p}{2\pi} = 1.6 \times 10^3 \ s^{-1}$

$$\therefore f_p \ll f_1, f_2$$

$$\therefore \Delta t \approx \frac{d}{c}\left(1 + \frac{\omega_p^2}{2\omega_1^2}\right) - \frac{d}{c}\left(1 + \frac{\omega_p^2}{2\omega_2^2}\right)$$

$$\Delta t = \frac{d\,\omega_p^2}{2c}\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right)$$

Ans: $\underline{d = 5.3 \times 10^{19} \, m}$

F. Dispersion in dielectrics

a simple, classical theory of dispersion – how ϵ, or equivalently n – depend on ω for low density dielectrics:

assume electrons are modelled as classical damped oscillators

$$m\ddot{x} + m\gamma_j\dot{x} + m\omega_j^2 x = qE_0 e^{j\omega t}$$

electron mass; strength of damping; restoring force on electron, ω_j: resonant frequency; electron charge; driving field at freq ω

let $x = x_0 e^{j\omega t}$

$$(\omega_j^2 - \omega^2 + j\omega\gamma_j)\, x_0 = \frac{qE_0}{m}$$

complex dipole moment $\tilde{p} = qx_0 = \frac{q^2 E_0}{m} \cdot \frac{1}{\omega_j^2 - \omega^2 + j\omega\gamma}$

complex polarization

$$\tilde{P} = \frac{q^2 N E_0}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 + j\omega\gamma_j} \quad (1)$$

weight of each contribution

no of atoms per unit volume

contributions from a given atom

$$= \epsilon_0(\tilde{\epsilon} - 1)E_0 = \epsilon_0(\tilde{n}^2 - 1)E_0$$

complex dielectric constant

complex refractive index

$$\therefore \tilde{n}^2 = 1 + \frac{q^2 N}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 + j\omega\gamma_j}$$

plasma result

γ, if 2^{nd} term is small, $(\tilde{n} \approx 1)$

$$\tilde{n} \approx 1 + \frac{q^2 N}{2\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 + j\omega\gamma_j} \equiv n_R - j n_I$$

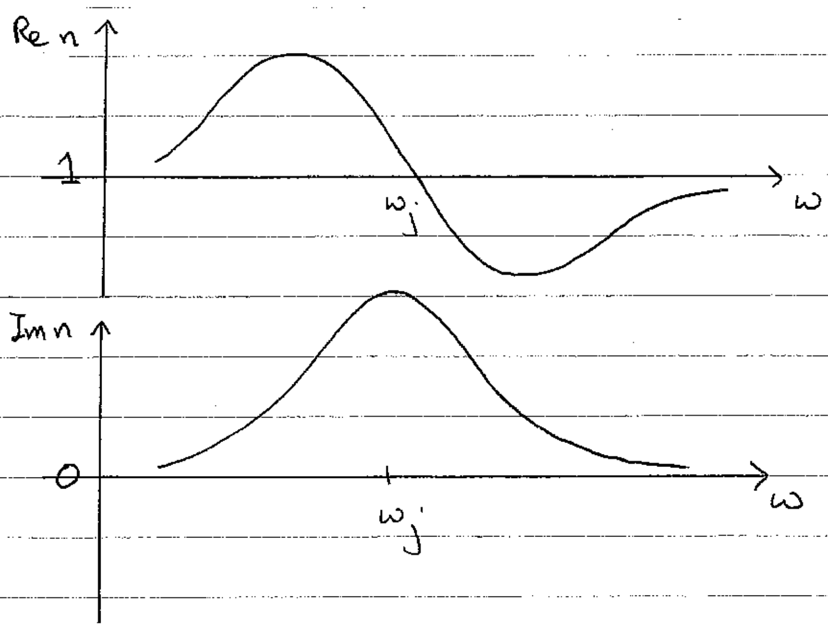
$$\boxed{\begin{aligned} n_R &\approx 1 + \frac{q^2 N}{2\epsilon_0 m} \sum_j \frac{f_j (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \omega^2 \gamma_j^2} \\ n_I &\approx \frac{q^2 N}{2\epsilon_0 m} \sum_j \frac{f_j \omega \gamma_j}{(\omega_j^2 - \omega^2)^2 + \omega^2 \gamma_j^2} \end{aligned}}$$

depends on γ_j - measure of energy lost as wave travels through material; we have modelled it as a damping term; physical processes are e.g. electronic transitions between energy levels; excitation of lattice vibrations ...

away from any resonance (ω_j) n_R decreases slowly with ω it is > 1 at low ω
can be < 1 at high ω
$\rightarrow 1$ as $\omega \rightarrow \infty$

normal dispersion

in the vicinity of a resonance ω_j



anomalous dispersion

P8

absorption coefficient in water

- from Jackson

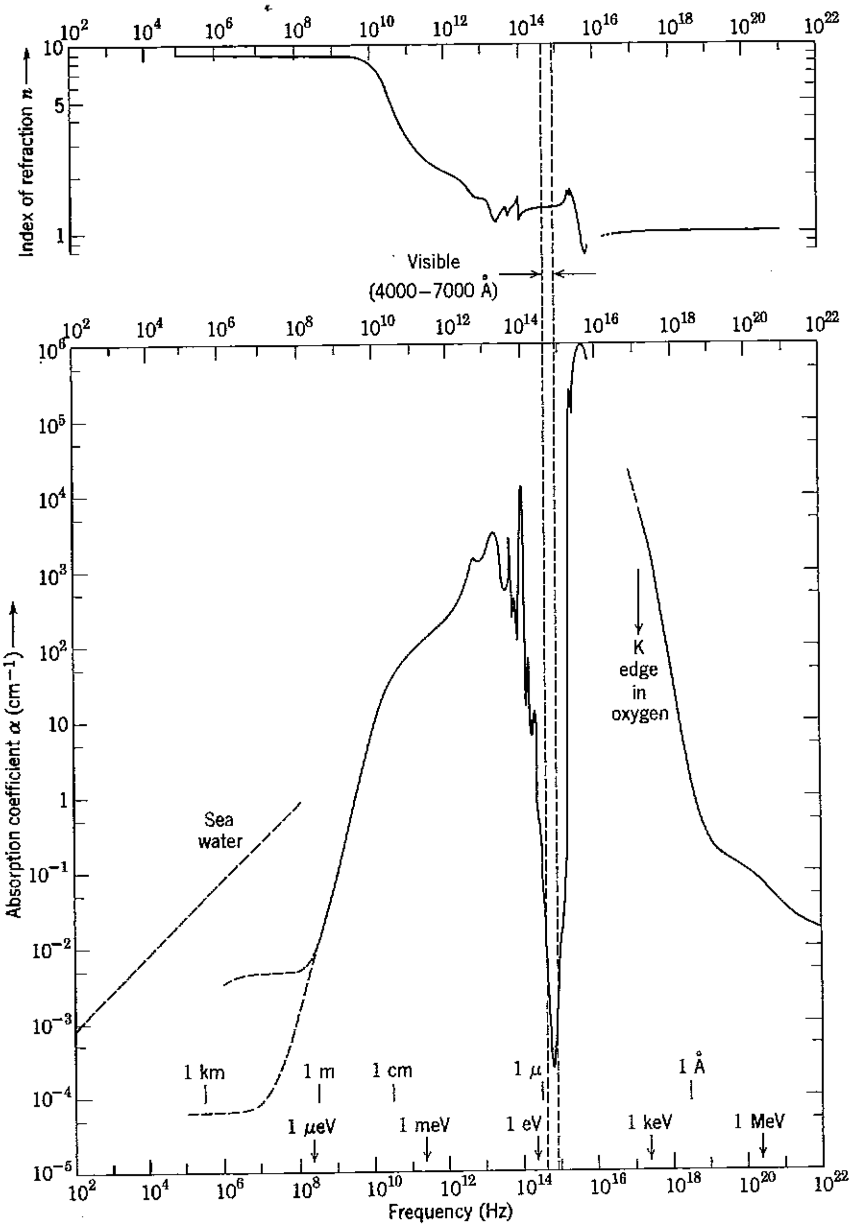
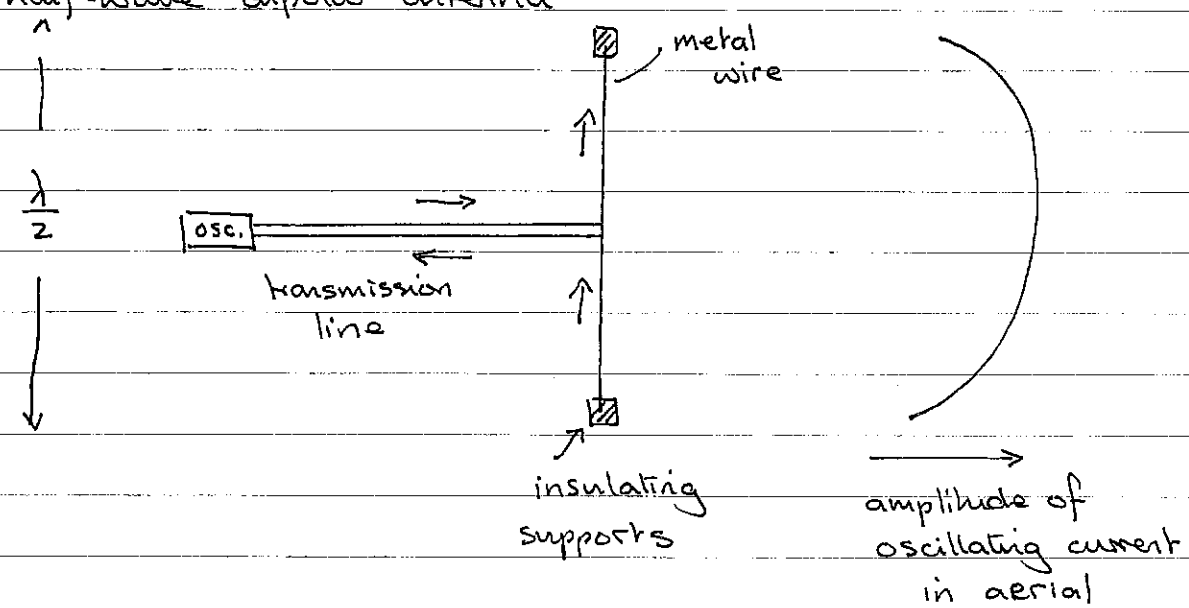


Fig. 7.9 The index of refraction (top) and absorption coefficient (bottom) for liquid water as a function of linear frequency. Also shown as abscissas are an energy scale (arrows) and a wavelength scale (vertical lines). The visible region of the frequency spectrum is indicated by the vertical dashed lines. The absorption coefficient for sea water is indicated by the dashed diagonal line at the left. Note that the scales are logarithmic in both directions.

G. Electric Dipole Radiation

e.m. fields are produced by accelerating charges and changing currents

half-wave dipole antenna



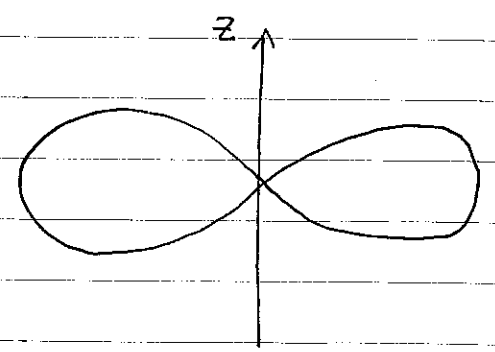
time averaged Poynting vector far from source

$$\underline{P} = \frac{\mu_0 p_0^2 \omega^4}{(4\pi)^2 c} \frac{\sin^2\theta}{2r^2} \hat{\underline{r}}$$

NB1 far from source

$\sim \frac{1}{r^2}$ i.e. energy transported to ∞

NB2 angular dependence of the intensity



$\sim \sin^2\theta$

need retarded potentials which account for $c \neq \infty$

— see next year.