I. Relation between Euclidean QFT and classical critical behaviour

Euclidean path integral for a scalar field theory in d dimensions:

Generating function: $$Z[J] = \int \mathcal{D}\phi \; e^{-\frac{1}{\hbar}\int\left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]d^dx + \int J\phi}$$

Source $J(x)$.

Functional derivatives $$\frac{1}{Z[0]} \frac{\delta^n}{\delta J(x_1)\ldots\delta J(x_n)} Z[J]\Big|_{J=0}$$

give correlation functions $\langle \phi(x_1)\ldots\phi(x_n)\rangle$.

$$\frac{\delta^n}{\delta J(x_1)\ldots\delta J(x_n)} \ln Z[J]\Big|_{J=0}$$ give connected correlation functions $\langle\phi(x_1)\ldots\phi(x_n)\rangle_c$

This all looks very much like classical statistical mechanics!

$$Z = \mathrm{Tr}\; e^{-\frac{1}{k_B T}E}$$

E.g. Ising model on a lattice : sites labelled by r

'spins' $s(r) = \pm 1$

external varying mag. field

$$E = -\frac{1}{2}\sum_{r,r'} \underbrace{J(r,r')}_{\text{exchange interaction}} s(r)\,s(r') - \sum_r \mu H(r) s(r)$$

Absorb $\frac{\mu}{kT}$ into H; $\frac{1}{kT}$ into J

$$Z_I[H] = \mathrm{Tr}_s \; e^{-\left(-\frac{1}{2}\sum K(r,r')s(r)s(r')\right) + \sum_r H(r)s(r)}$$

$$\frac{\partial^n}{\partial H(r_1)\ldots\partial H(r_n)} \ln Z[H] \text{ give } \langle s(r_1)\ldots s(r_n)\rangle_c.$$

Formally these look quite similar, ~~except~~: with

Euclidean QFT.		Stat mech
Action S	$\leftrightarrow$	Energy E
$\hbar$	$\leftrightarrow$	$k_B T$
	$\vdots$	

but there are differences, mainly:

1). continuous space r $\neq$ lattice r
2) continuous field $\phi(r)$ $\neq$ discrete spins $s(r)$.

To make the mapping more explicit, we can use the

<u>Hubbard-Stratonovich transformation</u> (which has many other uses)

This generalises the simple Gaussian integral

$$\int_{-\infty}^{\infty} d\phi \; e^{-\frac{1}{2}\phi K^{-1}\phi + \phi s}$$

$$= \int d\phi \; e^{-\frac{1}{2}(\phi - sK)K^{-1}(\phi - Ks) + \frac{1}{2}sKs}$$

$$= \int d\phi' \; e^{-\frac{1}{2}\phi' K^{-1}\phi' + \frac{1}{2}sKs}$$

$$= \sqrt{2\pi K} \; e^{\frac{1}{2}sKs}$$

to the case where K is a matrix, whose rows & columns are labelled by r, r':

$$\int \prod_r d\phi(r) \; e^{-\frac{1}{2}\sum_{r,r'} \phi(r) K^{-1}(r,r')\phi(r') + \sum_r \phi(r) s(r)}$$

$$\propto \sqrt{\det K} \quad e^{\frac{1}{2}\sum_{r,r'} s(r) K(r,r') s(r')}$$

Thus, apart from an unimportant constant,

$$Z_I[H] = \operatorname*{Tr}_{s} \int \prod_r d\phi(r)\, e^{-\frac{1}{2}\sum'_{r,r'} \phi(r) K^{-1}(r,r') \phi(r') + \sum'_r (\phi(r)+H(r)) s(r)}$$

The $\sum'_s$ can be done for each r:

$$\sum_{s(r)=\pm 1} e^{(\phi(r)+H(r))s(r)} \propto \cosh[\phi(r)+H(r)].$$

Thus

$$Z_I[H] = \int \prod_r d\phi(r)\, e^{-\frac{1}{2}\sum'_{r,r'} \phi(r) K^{-1}(r,r') \phi(r') + \sum'_r \ln\cosh[\phi(r)+H(r)]}$$

This is a <u>Euclidean lattice field theory</u>.

Usually $K(r,r') = K(r-r')$ (lattice translational invariance).

so if we write $$K(r-r') = \int_{BZ} \frac{d^d k}{(2\pi)^d} e^{ik(r-r')} \tilde{K}(k)$$

BZ ⇐ first Brillouin zone

Then $$K^{-1}(r-r') = - - - \frac{1}{\tilde{K}(k)}$$

and $$\frac{1}{2}\sum'_{r,r'} \phi(r) K^{-1}(r-r') \phi(r') = \frac{1}{2}\int_{BZ} \frac{d^d k}{(2\pi)^d} \tilde{\phi}(k)^* \frac{1}{\tilde{K}(k)} \tilde{\phi}(k).$$

Now suppose that $K(r-r')$ is <u>short-ranged</u>

This means that $$\tilde{K}(k) = \tilde{K}(0)\left[1 - R^2 k^2 + O(k^4)\right]$$

R ↑ <u>range</u> of interaction

$$\tilde{K}(k)^{-1} = \frac{1}{\tilde{K}(0)}\left[1 + R^2 k^2 + \dots\right].$$

Going back to real space & take "naive continuum limit"

$$\sum_r \to \int \frac{d^d r}{a^d} \qquad a: \text{lattice spacing.}$$

$$k^2 \to -\nabla^2$$

$$Z_I[H] \approx \int \mathcal{D}\phi \; e^{-\int \frac{d^d r}{a^d}\left[\frac{1}{2}\frac{1}{\tilde{K}(0)}\phi\,(1-R^2\nabla^2)\phi\right] + \int \frac{d^d r}{a^d} \ln\cosh[\phi+H]}$$

$\uparrow$ functional

[NB: $R \to \infty \Rightarrow \phi$: const. $\Rightarrow$ mean field theory.

Rescale $\qquad \phi^2 \to \tilde{K}(0)\,\phi^2\, a^d / R^2$

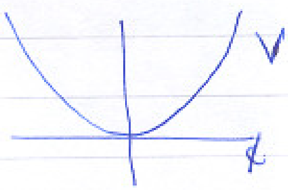
$$Z_I[H] \propto \int \mathcal{D}\phi \; e^{-\int d^d r \left[\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]}$$

where $$V(\phi) = \frac{1}{2R^2}\phi^2 - \frac{1}{a^d}\ln\cosh\left[\sqrt{\frac{\tilde{K}(0)a^d}{R^2}}\,\phi + H\right].$$

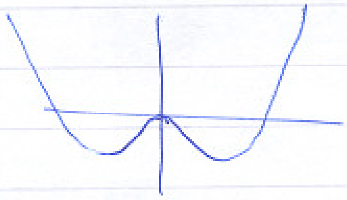
$$V(\phi)\Big|_{H=0} = \frac{1}{2R^2}\left(1-\tilde{K}(0)\right)\phi^2 + O\left(\frac{\tilde{K}(0)^2 a^d}{R^4}\right)\phi^4 + \dots$$

Recall $\tilde{K}(0) = \frac{1}{k_B T}\sum_r J(r)$

so for $T > T_{CMF} = \frac{1}{k_B}\sum_r J(r)$



$T < T_{CMF}$



We have cast the statistical mechanics model in the form of a Euclidean field theory.
In the naive continuum limit $a \to 0$ we simply replace sums by integrals. But we then recall that the Feynman diagram expansion has integrals which are ultraviolet divergent: this can be regularised with a cut-off $|k| < \Lambda$ on all loop integrals. The natural thing is to take $\Lambda \sim a^{-1}$ $\to$ thus the lattice spacing creeps back in

In order to take the continuum limit properly, we need to renormalise the theory.

Renormalised field theory and critical behaviour

Suppose we expand $V(\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{1}{4!}\lambda_0\phi^4 (+ \ldots$
and we drop the higher order terms, for the time being.

We can imagine computing the bare Green's functions (correlation functions)
$G_N = \langle\phi(x_1)\ldots\phi(x_N)\rangle$
in the cut-off theory with $|k| < \Lambda$. [These are the quantities most closely related to ~~stat.~~ original stat. mech. problem

G_N depends on m_0^2, λ_0, and Λ (as well as $(x_1 \ldots, x_N)$)

~~The statement~~

The statement of renormalisability (which varies slightly in detail depending on the theory and the renormalisation scheme) (for $d \leq 4$)
says that if we do field renormalisation: $\phi = Z_\phi^{1/2}\phi_R$

~~and~~ mass renormalisation $G_2 \underset{r\to\infty}{\sim} e^{-m_R \cdot r}$

and coupling constant renormalisation: $\lambda_R = \ldots$

so that Z_ϕ, m_R, λ_R are functions of m_0^2, λ_0 & Λ

then $(G_N)_R = \langle\phi_R(x_1)\ldots\phi_R(x_N)\rangle = Z_\phi^{-N/2}\langle\phi(x_1)\ldots\phi(x_N)\rangle$

when considered as a function of m_R, λ_R, has a finite limit as $\Lambda \to \infty$.

i.e. all the UV divergences have been absorbed into λ_R & m_R.

Historically in particle physics, this property of renormalisability was taken as essential: the cut-off Λ is introduced to define the theory, then removed: the renormalised Green's functions are the physical ones.

More recently, this point of view has changed slightly: all field theories are thought of as effective; valid only at energy (wavenumber) scales $\ll$ some cut-off Λ (e.g. the Planck scale). However since we would like the predictions of the theory to be insensitive to the particular value of Λ, ~~the~~ this is mathematically equivalent to the statement that the limit $\Lambda \to \infty$ is finite.

In statistical mechanics however, the physical correlation functions are the bare ones. What, then, is the role of the renormalised field theory? Its existence tells us that, when $m_R \ll \Lambda$, that apart from constant ~~per~~ factors of Z_ϕ, the correlation functions are universal = those of the renormalised theory, in the way they depend on $(x_1 \ldots, x_N)$.

m_R is just ~~the~~ inverse of ξ, the correlation length, so $m_R \ll \Lambda$ means $\xi \gg a$.

This happens close to a second order phase transition, and is called the scaling ~~reb~~ region.

From the existence of the limit $\Lambda \to \infty$, we can derive the Callan-Symanzik equation. For the renormalised Green functions do not know about Λ in this limit.

Thus
$$\Lambda \frac{\partial}{\partial \Lambda} G_{NR}\Big|_{\lambda_R, m_R} = \Lambda \frac{\partial}{\partial \Lambda}\left(Z_\phi^{-N/2} G_N\right)\Big|_{\lambda_R, m_R} = 0$$

G_N depends on λ_0, m_0 and Λ explicitly.

For simplicity let us work at the critical point where $m_R = 0$ $(\xi = \infty)$
Then m_0 is fixed in terms of (λ_0, Λ).
It is also useful ~~to~~ (recalling that $\lambda_0 \sim \dfrac{\tilde{K}(\epsilon)^2 a^d}{R^4}$)
~~so~~ to define $g_0 = \lambda_0 a^{4-d}$ as dimensionless coupling.

$$= \lambda_0 \Lambda^{d-4}$$

so G_N depends on (g_0, Λ) as well as $(x_1, \ldots, x_N)$

Then
$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g_0) \frac{\partial}{\partial g_0} - \frac{N}{2} \gamma(g_0) \right] G_N = 0$$

explicit

where
$$\beta(g_0) \equiv \Lambda \frac{\partial}{\partial \Lambda} g_0 \Big|_{\lambda_R \text{ fixed}}$$

$$\gamma(g_0) = \Lambda \frac{\partial}{\partial \Lambda} \ln Z_\phi \Big|_{\lambda_R \text{ fixed}}$$

This is an example of a renormalisation group equation:

if we change Λ (or equivalently a), it tells us how we have to change g_0 in order that G_N behaves simply.

If we change $a \to a e^{\ell}$ ($\ell > 0 \to$ coarse-graining)

Then
$$\frac{d}{d\ell} = a \frac{\partial}{\partial a} = -\Lambda \frac{\partial}{\partial \Lambda}$$

Thus
$$\frac{dg_0}{d\ell} = -\beta(g_0)$$

What is the form of $\beta(g_0)$? We know that

$$\lambda_R = \lambda_0 + O(\lambda_0^2)$$

In fact the second term is $O\left(\frac{1}{4-d}\right)$ reflecting a UV divergence for $d \geq 4$. Then

$$\lambda_R = \Lambda^{4-d}\left[g_0 + \frac{b}{4-d} g_0^2 + \ldots\right]$$

$\underline{b = O(1)}$

Take $\Lambda \frac{\partial}{\partial \Lambda}\Big|_{\lambda_R}$:

$$0 = (4-d)\left[g_0 + \frac{b}{4-d} g_0^2 + \ldots\right] + \left(1 + \frac{2b}{4-d} g_0 + \ldots\right)\beta(g_0)$$

CHECK

so
$$\beta(g_0) = -(4-d)g_0 + b g_0^2 + O(g_0^3). = \Lambda \frac{\partial g_0}{\partial \Lambda}$$

or
$$\frac{dg_0}{dl} = +(4-d)g_0 - bg_0^2 + \ldots$$

For $d \geq 4$ we see that $g_0 \to 0$: it is <u>irrelevant</u>

For $d < 4$ ~~then~~ $g_0 \to g^* = O(4-d)$ (if $b > 0$).

[NB. At this point we can see that the coeffs. of the terms we have dropped, like ϕ^6, $(\nabla^2\phi)^2$, etc are all irrelevant].

What does this mean for physics? (In this approach) we must also use dimensional analysis:

eg.
$$G_2(m_R^2 = 0, g_0, \Lambda, x_1, x_2) = \langle \phi(x_1)\phi(x_2) \rangle$$
$$= |x_1 - x_2|^{-(d-2)} F(|x_1 - x_2|\Lambda).$$

from rescaling $\phi \to \phi\sqrt{\frac{a^d}{R^d}}$

If we write $r = |\mathbf{x}_1 - \mathbf{x}_2|$ then

$$\left[r\frac{\partial}{\partial r} - \Lambda\frac{\partial}{\partial \Lambda} - (d-2) \right] G_2 = 0.$$

(Euler's eqn). So we can trade varying Λ for varying r, which is physically what we are interested in. We also see that ~~taking~~ increasing $a \Leftrightarrow$ decreasing $\Lambda \Leftrightarrow$ increasing r

so by taking $g \to g^*$ we explore the infrared $(r \to \infty)$ behaviour, appropriate for critical phenomena.

If we eliminate $\Lambda\frac{\partial}{\partial\Lambda}$ and take $g = g^*$, we get

$$\left[r\frac{\partial}{\partial r} - \gamma^* - (d-2) \right] G_2 = 0$$

where $\gamma^* = \gamma(g^*)$. The solution is $G_2 \propto \dfrac{1}{r^{d-2+\gamma^*}}$

γ^* is an example of a critical exponent (conventionally called η).

In field theory it is also called an <u>anomalous dimension</u>: what is really happening is that

$$\langle \phi(\mathbf{x}_1)\,\phi(\mathbf{x}_2) \rangle \sim \frac{1}{r^{d-2}} \left(\frac{a}{r}\right)^{\gamma^*}$$

$$\langle \phi_R(\mathbf{x}_1)\,\phi_R(\mathbf{x}_2) \rangle \sim \frac{1}{r^{d-2+\gamma^*}}$$

so it looks like ϕ_R has a dimension $\underbrace{\dfrac{d-2}{2}}_{\text{canonical}} + \underbrace{\dfrac{\gamma^*}{2}}_{\text{anomalous}} \equiv \underbrace{x_\phi}_{\text{scaling dimension}}$

General RG theory

We have seen in the above example of the Ising model (ϕ^4 theory in $d = 4-\epsilon$ dimensions) some of the essential features of an RG calculation in classical stat mech. (= euclidean QFT).

1). We vary the cut-off $\Lambda \to \Lambda e^{-\ell}$ (or $a \to a e^{\ell}$) (corresponding to coarse graining - eliminating short wavelengths) and attempt to vary the dimensionless couplings (like g_i) so as to keep the long-wavelength physics 'the same' [this might require eg field renormalisation]

2). RG equations $\quad \dfrac{dg_i}{d\ell} = -\beta_i(\{g_j\})$

3). Look for fixed points where $\beta_i(\{g_j^*\}) = 0$.

4) Linearise around these $\quad g_i' = g_i - g_i^*$ (might require a rotation)

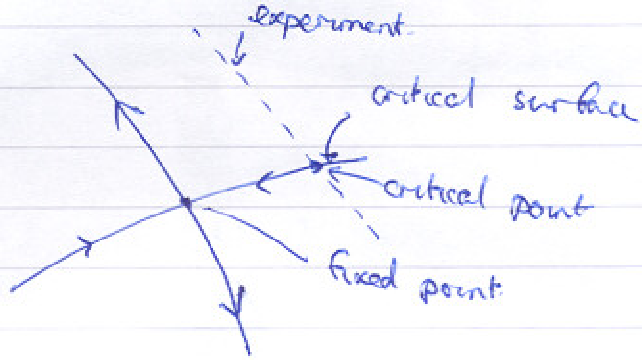
such that $\quad \dfrac{dg_i'}{d\ell} = y_i g_i' + O(g'^2)$.

5) RG eigenvalues:

$y_i > 0 \quad g_i'$ relevant
$y_i < 0 \quad g_i'$ irrelevant
($y_i = 0 \quad$ marginal).

6). Assume local RG flows extend to some finite region:

Eg



This accounts for universality: many different systems described by same fixed point.

7). Behaviour of correlation functions ~~near~~ at fixed point

$$S = S_{\text{fixed pt.}} + \sum_i g_i' \int \frac{d^d r}{a^d} \underset{\substack{\uparrow \\ \text{scaling operators}}}{\phi_i(r)}$$

$$(\text{e.g. } g = H \quad \phi \propto s).$$

If under $a \to a e^{\ell}$ $\quad g_i' \to e^{y_i \ell} g_i'$

then $\phi_i \to e^{(d - y_i)\ell} \phi_i$

But $\langle \phi_i(r_1) \phi_i(r_2) \rangle = \left(\frac{a}{r}\right)^{2x_i} \Rightarrow$

so $\boxed{x_i = d - y_i}$.

This relates correlation functions at the fixed pt. to critical exponents away from the fixed pt.

E.g. suppose that for $g' \neq 0$ the theory develops a finite correlation length ξ.

Dimensional analysis: $\quad \xi = a f(g')$.

Under ~~renorm~~ RG we keep ξ (long distance physics) the same:

so $\quad a e^{\ell} f(g' e^{y\ell}) = a f(g')$

$$\Rightarrow f(g') \propto g'^{\,1/y} \quad \Rightarrow \xi = a g'^{\,\nu}$$

$\underline{\nu = 1/y}$.