Field Theory in Condensed Matter Hilary 2002 Prof. J. Cardy

Problem Set 2

The solutions to these problems should be handed in by **Monday, April 22, 5 pm,** for marking. They should be put in my mailbox in Theoretical Physics. They will be reviewed at a problems class, to be announced, early in Trinity Term. Any questions or clarifications before the due date may be addressed to me at cardy@thphys.ox.ac.uk.

1. This question concerns the $2d$ XY model.

   a) Calculate the coefficient A in the RG equation $dK/d\ell = Ay^2 + \cdots$, (where K is the usual spinwave stiffness of the gaussian action $S_G = \frac{1}{2}K\int(\nabla\theta)^2 d^2r$, and y is the vortex fugacity,) by computing the appropriate OPE coefficient. [You can do this by comparing the correlation functions $\langle(\nabla\theta(r'))^2 V(r_1)\overline{V}(r_2)\rangle$ and $\langle(\nabla\theta(r'))^2\langle(\nabla\theta(r))^2\rangle$ as r_1 and r_2 approach r. You should average over orientations of the vortex-antivortex pair.]

   b) Suppose the gaussian action above is also perturbed by a term $y_p\int\cos(p\theta(r))d^2r$. Work out the lowest order RG equations, analogous to the Kosterlitz equations, for $dy_p/d\ell$ and $dK/d\ell$.

   c) Now suppose that both perturbations of the gaussian action are present. Show that, for p larger than a critical value p_c, part of the gaussian line of fixed points is stable to both kinds of perturbation. What does this mean for the phase diagram of such a model versus temperature? What happens when $p = p_c$?

2. Two spinless Luttinger liquids, each with identical parameters v_F and g_2, and labelled by indices (1) and (2), are coupled together by (a) a hopping term

$$t\int\left(\psi_L^{(1)\dagger}\psi_L^{(2)} + \text{h.c.} + \psi_R^{(1)\dagger}\psi_R^{(2)} + \text{h.c.}\right)dx$$

   and (b) scattering terms

$$\int g_4'\left(\rho_L^{(1)}\rho_L^{(2)} + \rho_R^{(1)}\rho_R^{(2)}\right) + g_2'\left(\rho_L^{(1)}\rho_R^{(2)} + \rho_R^{(1)}\rho_L^{(2)}\right)dx$$

   a) Write the bosonised form of the complete hamiltonian in its full glory, in terms of fields $\phi_{L,R}^{(1,2)}$.

b) Assuming that the hopping term t is irrelevant, show that the bosonised hamiltonian can be written in terms of two decoupled Luttinger liquids depending on fields $\phi_{L,R}^{(+,-)} = \phi_{L,R}^{(1)} \pm \phi_{L,R}^{(2)}$. Deduce the form of the single-particle density of states as $\omega \to 0$.

c) Under what conditions is the assumption that t is irrelevant valid? If it is not valid, what happens?

3. An assembly of bosons, at chemical potential μ, with short-range repulsive interactions, is described by a complex field $\hat{\Psi}(x)$ and a quantum hamiltonian

$$\hat{H} = \int [\frac{1}{2}(\nabla\hat{\Psi}^\dagger)(\nabla\hat{\Psi}) + \frac{1}{4}\lambda_0(\hat{\Psi}^\dagger)^2\hat{\Psi}^2]d^dr$$

Here $[\hat{\Psi}(r), \hat{\Psi}^\dagger(r')] = i\hbar\delta^{(d)}(r-r')$, and $\lambda_0 > 0$. The number operator is $\hat{N} = \int \hat{\Psi}^\dagger\hat{\Psi}\, d^dr$.

a) Considering this first as a classical hamiltonian, what is the lagrangian (in imaginary time)? Hence write the grand partition function $\Xi = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}$ as a functional integral over c-number fields $\Psi(t,r)$ and $\Psi^*(t,r)$.

b) Write down the Feynman rules for this dynamical field theory, at zero temperature. Show that there are no loop corrections to the one-particle propagator, and that the loop corrections to the renormalisation of λ take a simple form. Hence compute the beta-function of this theory, when $\mu = 0$, exactly. What is the upper critical dimension d_c?

c) Show that, still at zero temperature, the theory has a quantum phase transition at $\mu = 0$. What is the nature of the ground state when $\mu < 0$ and when $\mu > 0$? What happens when $d \leq d_c$?

d) Now consider finite temperature. Decompose the fields into their Matsubara components, and write down the effective field theory for the zero-frequency component. In what universality class is the transition at finite temperature? Show, using dimensional analysis, that the critical value μ_c depends on the temperature as $\mu_c \propto T^{2/(4-d)}$, for $d < 4$. Sketch the phase diagram as a function of μ and T, indicating crossovers where appropriate, for different values of d.

4. The charge density $\rho(t,x)$ in a fast ionic conductor in a constant uniform external electric field $\mathbf{E}$ satisfies the Langevin equation

$$\partial\rho/\partial t = -\mathbf{E}\cdot\nabla(\rho^2) + \nu\nabla^2\rho + \zeta$$

where $\langle\zeta(t,x)\zeta(t',x')\rangle = -D\nabla^2\delta(x-x')\delta(t-t')$. [Note that this equation conserves the total electric charge, and is invariant under simultaneously changing the signs of ρ and $\mathbf{E}$.]

a) Show that, as with the KPZ equation, the nonlinear term cannot be written as a derivative of some effective free energy, so there is no detailed balance or FDT.

b) Write down the field theory action using the response field formalism. Write down the Feynman rules, and do dimensional analysis to work out the dimensionless effective coupling constant and the upper critical dimension.

c) Show that, for $d=1$, the model is the same as KPZ, by the substitution $\rho = \partial_x h$. Hence deduce the dynamic scaling form for the correlation function $\langle\rho(t,x)\rho(0,0)\rangle = |x|^{-(d-2+\eta}\Phi(t/x^z)$, and the exact values of the exponents η and z, for $d=1$.

d) Show that the arguments (i.e. Galilean invariance plus existence of a simple invariant measure) which lead to these exact values for the $d=1$ KPZ equation work for *any* value of $d<d_c$ for this problem. Deduce the exact values of the exponents η and z for all $d<d_c$.