Field Theory in Condensed Matter Hilary 2002 Prof. J. Cardy

Problem Set 1

The solutions to these problems should be handed in by **Tuesday Feb 19, 5 pm,** for marking. They should be put in my mailbox in Theoretical Physics, or handed to me after the lecture. They will be reviewed at the problems class on **Mon Feb 25** at 2 pm in the NAPL Fisher Room. Any questions or clarifications before the due date may be addressed to me at cardy@thphys.ox.ac.uk.

1. [The purpose of this question is to test your understanding of simple scaling/RG arguments. It should not involve any complicated explicit calculations.] The Ising model in d dimensions, away from its critical temperature, is described by the fixed point action perturbed by $t\int \phi_2 \delta^d r$, where $\phi_2 =: \phi^2 :$ and $t \propto T - T_c$. Assume that ϕ_2 has scaling dimension x_2. What is the relation between x_2 and the critical exponent ν describing the behaviour of the correlation length $\xi \sim t^{-\nu}$? Write down a scaling form for the correlation function $\langle \phi_2(r)\phi_2(0)\rangle$ away from the critical point as a function of r and ξ. The integral of this correlation function over r is proportional to the specific heat C [why?], which is supposed to behave like $t^{-\alpha}$. Hence work out a relation between ν and α. Summarise what were the crucial assumptions involved in your argument, and try to explain why they go wrong for dimensions $d > 4$.

The following questions should be answered using the 'poor man's' perturbative RG:

2. The tricritical Ising model is described by a field theory with action

$$S = \int [(\nabla\phi)^2 + g_2\phi_2 + g_4\phi_4 + g_6\phi_6] d^d r$$

where $\phi_n =: \phi^n :$. Work out the OPE coefficients at the gaussian fixed point and hence write down the RG equations to second order in the g's. Show that there is a fixed point in $d = 3 - \epsilon$ dimensions at which $g_6 = O(\epsilon)$ and the other g's are smaller. Calculate the RG eigenvalues of g_2' and g_4' to first order in ϵ.

3. The Heisenberg model is described by a 3-component field $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ and action

$$S = \int [(\nabla\vec{\phi})^2 + g_2\phi_2 + g_4\phi_4] d^d r$$

where now $\phi_{2n} =: ((\vec{\phi})^2)^n :$. Show that there is a fixed point in $4 - \epsilon$ dimensions where $g_4^* = O(\epsilon)$ and $g_2^* = O(\epsilon^2)$. [Note: from your experience with the Ising model you should be able to see that you need compute only *one* OPE coefficient to verify this.]

Now suppose you add a perturbation $g' \int \phi' d^d r$ where $\phi' = 2s_3^2 - s_1^2 - s_2^2$. [What is special about the way this particular combination transforms under O(3) rotations?] Work out the RG eigenvalue of g' to $O(\epsilon)$.

4. Consider a system made of two layers coupled together. Each layer is a two-dimensional system described by a reduced hamiltonian (action) $S_j = S_j^* - g_j a^{x_j - 2} \int \phi_j(r) d^2 r$ where $j = 1, 2$ labels the two layers. For simplicity, each fixed point action S_j^* is perturbed by only one scaling operator ϕ_j, with scaling dimension x_j. Initially the layers are decoupled, and the OPEs have the form

$$\phi_j \cdot \phi_j = \mathbf{1} + b_j \phi_j + \cdots$$

where b_j is an OPE coefficient. The two layers are now coupled by adding a term $-\Delta a^{x_1 + x_2 - 2} \int \phi_1(r) \phi_2(r) d^2 r$ to the action. Work out the RG equations to second order in g_2, g_2 and Δ, and explore all the possible fixed points. Assume that $\epsilon \equiv 2 - x_1 - x_2$ is small.