Problem Set

The solutions to these problems should be handed in by **Wednesday, January 14** for marking. They should be sent to Dr. P. Calabrese in Theoretical Physics. They will be reviewed at the problems class on **Wednesday, January 21** at 2 pm in the Fisher Room, which will be conducted by Dr. Calabrese. Any questions before the due date may be addressed to me at j.cardy1@physics.ox.ac.uk, or, after Jan. 5, to Dr. Calabrese at calabres@thphys.ox.ac.uk

Parts of questions marked † might be found slightly harder.

1) A one-dimensional harmonic oscillator has an action $S = \int(\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2)dt$. Note this looks like the action for a quantum field theory in zero space and one time dimension.

   (a) Use this observation to write down a formula for the ground state expectation value of the time-ordered product $\langle 0|\mathbf{T}(q(t_2)q(t_1))|0\rangle$ in terms of the Feynman propagator. Check your result when $t_2 = t_1$ against the standard result for $\langle q^2\rangle$ using the ground state wavefunction.

   (b) Now work out the propagator in imaginary time τ with periodic boundary conditions $q(\tau + \beta\hbar) = q(\tau)$. (It is the same calculation as in the lecture, but now the Fourier modes of $q(\tau)$ are discrete.) Hence derive a formula for $\langle q^2\rangle$ at finite temperature $kT = \beta^{-1}$. Check that this agrees with the result of classical statistical mechanics at high enough temperature.

   (c) Now think of $q(\tau)$ as the displacement of a long linear polymer, with τ being a coordinate along the length of the polymer. It is held in place by forces $\pm f$ acting at the points τ_1 and τ_2. The energy is

$$E = \int(\tfrac{1}{2}(dq/d\tau)^2)d\tau + f(q(\tau_1) - q(\tau_2))$$

   which is the same as the Euclidean action above, with $\omega = 0$, and a source term $J(x) = f(\delta(\tau - \tau_1) - \delta(\tau - \tau_2))$. Use the results in class for the generating function $Z[J]$ to calculate the classical partition function for the polymer at temperature T.

2) What are the propagators (in momentum space) of the euclidean field theories with the following free actions? (You should be able to write these down more or less by inspection. There may be more than one propagator if the field has more than one component.)

a) $S_0 = \frac{1}{2}\int dxdydz[(\partial_x\phi)^2 + (\partial_y\phi)^2 + \alpha(\partial_z^2\phi)^2 + m^2\phi^2]$;

b) $S_0 = \frac{1}{2}\int d^dx[(\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m_1^2\phi_1^2 + m_2^2\phi_2^2 + 2\lambda\phi_1\phi_2]$;

c) † $S_0 = \int d\tau d^Dx[\frac{1}{2}(\phi^*\partial_\tau\phi - \phi\partial_\tau\phi^*) + \lambda(\nabla\phi^*)\cdot(\nabla\phi)]$ (ϕ is complex).

3) What are the vertices corresponding to the following interactions in a euclidean field theory? Draw the way they would appear in a Feynman diagram, give any numerical factors, and label the lines where appropriate.

a) $\lambda(\phi^{*3} + \phi^3)$ (ϕ is complex); b) $\lambda\cos(\beta\phi)$; c) $\lambda\phi^2\,(\partial_\mu\phi)(\partial^\mu\phi)$.

4) Consider a theory with one complex scalar field ϕ and a real scalar field Φ, with a lagrangian density

$$(\nabla\phi^*)(\nabla\phi) + m^2\phi^*\phi + \tfrac{1}{2}((\partial\Phi)^2 + M^2\Phi^2) + \lambda\phi^*\phi\Phi$$

a) *Write down* the Feynman rules for this theory (be careful to use a different sort of line for each propagator). Draw the diagrams, to second order in λ. which contribute to the Fourier transforms of the one-particle irreducible connected parts of the correlation functions $\langle\phi\phi^*\rangle$, $\langle\Phi\Phi\rangle$, $\langle\phi\phi\phi^*\phi^*\rangle$ and $\langle\phi\phi^*\Phi\Phi\rangle$, and give their explicit expressions together with symmetry factors.

b) Consider the elastic scattering of two ϕ-particles in this theory, in Minkowski space. Give an explicit expression for the $\hat{T}$-matrix and hence the elastic cross-section, to $O(\lambda^2)$, in terms of the centre-of-mass energy and the scattering angle (you may use the result quoted in the hand-out.)

5) Consider a real scalar field theory with a negative (mass)^2 term, with a (euclidean) lagrangian density

$$\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{4!}\lambda\phi^4$$

(such a theory describes an Ising magnet below its Curie temperature). Why does the straightforward perturbation theory in λ go wrong? Show that it is possible to construct a sensible expansion by first suitably shifting ϕ = constant + ϕ', so that there are no linear terms in ϕ'. *Write down* the Feynman rules for this shifted theory, and draw the one loop diagrams which now contribute to the two-point connected correlation functions.

6) Consider the theory of a single scalar field with lagrangian density

$$\tfrac{1}{2}((\partial\phi)^2 + m_0^2\phi^2) + \frac{1}{4!}\lambda_0\phi^4 + \frac{1}{6!}u_0\phi^6.$$

Do dimensional analysis to find the dimension d_c in which this theory is exactly renormalisable, that is, superficial divergences occur in only a finite number of $\Gamma^{(N)}$, but to all orders in perturbation theory. In this number of dimensions, how many renormalisations are required (mass, field, etc.) to render the theory finite? †Calculate the beta function for u_R at the point where $m_R = \lambda_R = 0$, to lowest nontrivial order, and show that it has an infrared stable zero in $d_c - \epsilon$ dimensions. [You will meet one Feynman integral. ††† if you manage to evaluate it, otherwise assume it has the form C/ϵ where C is a constant.]

7) The so-called Lifshitz point of a magnet is described by a real scalar field theory with a $(\lambda/4!)\phi^4$ interaction, but with an inverse bare propagator $r_0^2 k_1^4 + \mathbf{k}_\perp^2 + m^2$, where $\mathbf{k}_\perp$ is a $(d-1)$-dimensional momentum vector. The theory is anisotropic between the 1-direction and the other $(d-1)$ directions. r_0 is a parameter. Treating k_1 and $k_\perp$ as having different, independent, dimensions, work out the engineering dimensions of the vertex functions $\Gamma^{(N)}$. What is the critical dimension d_c of this theory? †Which quantities need to be renormalised to make the theory finite at d_c?

8) In QCD, the renormalisation group functions have the form $\beta(g_R) = -Bg_R^3 + O(g_R^4)$ and $\gamma(g_R) = Cg_R^2 + O(g_R^3)$, where B and C are positive constants.

a) QCD is believed to exhibit dynamical mass generation, which means that although there is no mass term in the Lagrangian, nevertheless the particles in the theory (hadrons) have non-zero mass. By observing that the mass of, e.g., the proton must, on dimensional grounds, have the form $M = \mu\, f(g_R)$, but that M cannot in fact depend on the arbitrary renormalisation scale μ, deduce a relation between $f(g_R)$ and $\beta(g_R)$. Using the form of $\beta(g_R)$ from above, deduce how M depends on g_R for small g_R.

b) †What is the asymptotic behaviour of $\Gamma^{(2)}(p)$ as $p \to \infty$? (You must use the full solution of the Callan-Symanzik equation in this case. Note that 'asymptotic behaviour' means not just that it goes to infinity, but how it goes to infinity.)