Conformal Invariance and Statistical Mechanics

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1. Introduction

1.1 Outline and aims

In the last few years, there has been an explosion of interest and of progress in two-dimensional conformal field theory. The motivation for studying this subject is three-fold: (1) for its intrinsic mathematical interest, which is fascinating in itself, and which reveals deep connections with other diverse branches of mathematics; (2) because conformal field theories correspond to ground states of string theory; and (3) because statistical mechanics models at a critical point are conformally invariant on large distance scales. The questions to be asked in conformal field theory are thus dependent on ones point of view, although it is rather easy to forget this when following up an interesting idea. As a result, this subject has provided some of the most prolific examples of interdisciplinary cross-fertilization in theoretical physics.

In these lectures, I want to approach the subject from the third point of view. We shall therefore always try to have in mind the questions that statistical mechanics tries to answer when critical phenomena are discussed. These include (a) the classification of universality classes, and, equally important, the development of criteria to decide whether a given system is in a particular universality class; (b) the calculation of critical exponents; (c) the calculation of other universal amplitudes, including finite-size effects at criticality, and ratios of thermodynamic quantities away from criticality; and (d) the calculation of correlation functions, both at and away from criticality.

As with any branch of theoretical physics, the ultimate test of these results is experiment. Unfortunately, precise real experiments on critical phenomena in two dimensions are notoriously difficult. In addition, the number of systems available to the experimentalist is small. While there was a great deal of excitement ten years ago in comparing the predictions of the Ising model and the 3-state Potts model to experimental data on adsorbed systems, the new models discovered since then have found little practical application.* As a result, nearly all the tests of the predictions of conformal field theory have been against numerical experiments. While these tests have been stringent and successful in confirming the concepts of conformal invariance, the subject could definitely benefit from closer contact with real experiments.

It would be impossible to cover even the essentials of this rapidly growing field in nine lectures. I have therefore made some rather arbitrary selections, partly dictated by my own interests and (lack of) expertise in various areas. I shall assume some of the basics of conformal field theory *à la* Belavin, Polyakov and Zamolodchikov (BPZ) [1] which will be covered in Ginsparg's lectures. However, I shall endeavor to state clearly the results which I am using. Many of the basic ideas I shall cover are independent of the value of c. When I want to specialize, for reasons of brevity I shall restrict myself to theories with $c \leq 1$. Although there has been a lot of recent work on theories with $c > 1$, much of it is modelled on the more complete results for $c < 1$, which therefore form a good starting point.

1.2 Statistical mechanics and quantum field theory

Let us begin by reviewing the connection between these two subjects, with the partial aim of avoiding any confusion in terminology later on. In classical statistical mechanics, one of the objects of interest is the partition function

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\mathcal{H}/kT}, \tag{1.1}$$

where $\mathcal{H}$ is the classical Hamiltonian. We shall usually absorb the factor of $(kT)^{-1}$ into $\mathcal{H}$. The analogous quantity in Euclidean quantum field theory is the generating function

$$Z = \int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]/\hbar}, \tag{1.2}$$

where S is the action, depending on a set of local fields $[\phi]$. The obvious similarity between these expressions deepens when we realize that (1.2) must be defined through some kind of

* The spin chains to be discussed in Affleck's lectures form a notable exception.

regularization procedure of the short-distance behavior, and that a natural way to do this is to put the quantum field theory on a lattice, with some microscopic spacing a. Although quantum field theories have continuous fields as degrees of freedom, while statistical mechanics systems usually involve discrete quantities like Ising spins, this difference becomes irrelevant close to a critical point. This idea is based on the concept of universality and its underlying framework, the renormalization group. This states that when the correlation length $\xi \gg a$, the behavior of the correlation functions over distances much larger than a is independent of the microscopic details of the Hamiltonian or action, whether it describe a discrete statistical mechanics system or a lattice quantum field theory, and that these correlation functions are identical to those of a corresponding continuum quantum field theory, when suitably normalized. Thus the problem of understanding all possible universality classes is equivalent to that of classifying possible continuum quantum field theories. As we shall see, conformal invariance strictly speaking applies only to continuum theories, and so its applications to statistical mechanics are inevitably framed in the language of quantum field theory.

In the continuum limit, the correlation functions contain no dependence on the microscopic distance a. Exactly at a critical point, the correlation length ξ is infinite, and the corresponding quantum field theory is massless. The local renormalized fields are usually referred to as scaling operators. This terminology is a little confusing, because in Euclidean quantum field theory they are not operators, but only become so when we go to a Hilbert space formalism for the field theory. The 2-point function of a renormalized operator which is a scalar under rotations has the power law behavior

$$\langle\phi(r)\phi(0)\rangle = (\kappa|r|)^{-2x}, \tag{1.3}$$

where x is the scaling dimension of ϕ. This equation fixes the normalization of ϕ. The inverse length κ is the usual arbitrary length scale which must be introduced to define a renormalized massless theory. At the risk of some equations not looking dimensionally correct, we shall take $\kappa = 1$. The renormalized operators are related to those defined on the lattice by

$$\phi(r) \sim a^x \phi_{\mathrm{lattice}}(r), \tag{1.4}$$

up to some non-universal constant of order unity. The meaning of (1.4) is that correlation functions of either side are asymptotically equal for distances $\gg a$.

Scaling operators may have non-trivial properties under rotations, in which case (1.3) is modified. In two dimensions, where we shall be living exclusively in these lectures, this modification is easy: in complex co-ordinates $(z, \bar{z})$

$$\langle \phi(z, \bar{z}) \phi(0, 0) \rangle = z^{-2h} \bar{z}^{-2\bar{h}}, \tag{1.5}$$

where $x = h + \bar{h}$ is the scaling dimension, and $s = h - \bar{h}$ is the spin of ϕ. The real numbers $(h, \bar{h})$ are called (confusingly) the complex scaling dimensions of ϕ. In string theory, they are referred to as the conformal weights. Note that $\bar{h}$ is not necessarily the complex conjugate of h.

A very useful concept in statistical mechanics is that of the transfer matrix. It is easiest to define this by an example. Consider an Ising model on a square lattice. Take a slice of the lattice from the row $y = y_0$ to the row $y = y_0 + \Delta y$, and, fixing the spins on these two rows to assigned values, calculate the partition function of the slice. The result is an element of the transfer matrix $\hat{T}$, whose rows are labelled by the state of all the spins along $y = y_0$, and whose columns are labelled by the state of the spins along $y = y_0 + \Delta y$. Usually, Δy is one or a few lattice spacings. The transfer matrix is an operator acting on a Hilbert space, on which the other local quantities which become scaling 'operators' $\phi(x, y)$ in the continuum limit act like *bona fide* operators $\hat{\phi}(x)$. The partition function Z in (1.1) can now be calculated by raising $\hat{T}$ to the appropriate power and taking the trace.

If we now dissect the path integral in (1.2) in the same way, we are doing nothing but following the standard route from the path integral formulation of a quantum field theory to the Hamiltonian formulation, quantized on constant 'time' surfaces $y =$ const. Thus we see that, in the continuum limit, the quantum Hamiltonian is essentially the logarithm of the transfer matrix:

$$\hat{T} = \mathrm{e}^{-\Delta y \hat{H}}. \tag{1.6}$$

We shall try to avoid confusing the classical statistical mechanics Hamiltonian $\mathcal{H}$ and the quantum Hamiltonian $\hat{H}$ for the quantum field theory, by referring to the former as the action, and denoting it by S.

Note that we could have sliced up the plane in other ways to arrive at a Hilbert space description. The notions of slicing in polar co-ordinates (r, θ) parallel to the curves $r =$ const. or $\theta =$ const. will arise elsewhere in these lectures, corresponding to the radial quantization of conformal field theory, and the corner transfer matrix of Baxter, respectively.

The power law behavior (1.3) of the correlation functions corresponds to the transformation law $\phi(r) \to b^x \phi(r')$ under a dilatation, or scale transformation, $r \to r' = b^{-1}r$. This non-trivial behavior arises because of the existence of the microscopic length a needed to regularize the theory. One way to picture the physical significance of the co-ordinate change $r \to r'$ is to imagine the lattice as imposing a co-ordinate grid on the plane. Under a renormalization group transformation, a is increased to ba, and the short-wavelength degrees of freedom are integrated out of the partition function, in such a way as to leave the large distance physics unchanged. In doing this, the action S will be modified, unless it happens to lie at a renormalization group fixed point S^*. Thus the co-ordinate change $r \to r'$ is not simply a relabelling of the points of the plane, but carries physical implications.

Figure 1.1 Generalized RG transformations corresponding to (a) a scale transformation (b) a conformal transformation (c) a general co-ordinate transformation. In each case the new lattice gives a grid with respect to which the co-ordinates r' of a given point are defined.

Let us now consider a general infinitesimal co-ordinate transformation $r^\mu \to r^\mu + \alpha^\mu(r)$. There is no reason for the action to remain invariant under such a transformation, even if it initially lies at a fixed point. Assuming that the change depends in a local way on the function $\alpha^\mu(r)$, the most relevant term which can enter depends on the derivatives $\partial^\mu\alpha^\nu$:

$$\delta S = -\frac{1}{2\pi}\int T_{\mu\nu}(r)\,\partial^\mu\alpha^\nu(r)\,d^2r. \tag{1.7}$$

This defines the *stress tensor*, which is the Euclidean version of the (improved) energy-momentum tensor. The factor of $(2\pi)^{-1}$ is put in to avoid such factors elsewhere in the calculations. Eq. (1.7) has a simple physical interpretation: the interactions of the statistical mechanics model give rise to an effective elasticity of the medium. The contribution of this to the Hamiltonian (action) is simply the integral of the stress tensor contracted with the strain tensor $\partial^\mu\alpha^\nu$.

In a Hilbert space formalism, the quantum Hamiltonian $\hat{H}$ is related to the integral of the 'time-time' component of $T_{\mu\nu}$ over the space-like curve on which we choose to quantize the theory. For example, in the usual row-to-row transfer matrix formalism discussed in Sec. (1.2),

$$\hat{H} = \frac{1}{2\pi}\int \hat{T}_{yy}(x)dx. \tag{1.8}$$

To derive this, consider the response of the transfer matrix acting from row $y = y_1$ to row $y = y_3$ to a co-ordinate change $x' = x$, $y' = y + \alpha H(y - y_2)$, where $H(y - y_2)$ is a step function at y_2, and $y_1 < y_2 < y_3$. Eq. (1.8) is of course the Euclidean version of the familiar quantum field theory statement that the Hamiltonian is the space integral of the time-time component of the energy-momentum tensor.

The tensor $T_{\mu\nu}$ may be decomposed into an antisymmetric piece, a diagonal piece, and a traceless symmetric piece. The former will always vanish if the system is rotationally invariant. We shall always assume this is the case. Systems in statistical mechanics are of course usually defined on a lattice, but in most cases rotational invariance is regained at large distances, because the operators which break this symmetry are irrelevant in the renormalization group sense. In some cases, the lattice theory may have a trivial anisotropy which may be removed by an anisotropic rescaling of distances. An example is the Ising model on a square lattice with unequal couplings in the horizontal and vertical directions. In some other models, on the other hand, the correlation lengths in different directions actually diverge with different exponents as we approach criticality. These models are intrinsically anisotropic, and we shall have nothing to say about them.

The trace $\Theta = T^\mu_\mu$ must vanish at a critical fixed point, because then $\delta S = 0$ under a dilatation where $\alpha^\mu \propto r^\mu$. However, this statement, together with the locality of $T_{\mu\nu}$, also implies the vanishing of δS under all transformations such that the traceless symmetric part of $\partial^\mu \alpha^\nu$ vanishes. Such tranformations correspond locally to a rotation plus a dilatation. This is the set of *conformal* transformations. In complex co-ordinates, $\partial_{\bar{z}} \alpha^z = \partial_z \alpha^{\bar{z}} = 0$. Thus α^z and $\alpha^{\bar{z}}$ are respectively analytic and antianalytic.

In two dimensions, complex co-ordinates are very useful for working with the stress tensor. In general, $T_{\mu\nu}$ has three independent components $T \equiv T_{zz}$, $\overline{T} \equiv T_{\bar{z}\bar{z}}$ and $\Theta = 4T_{z\bar{z}} = 4T_{\bar{z}z}$. The conservation law $\partial^\mu T_{\mu\nu} = 0$ reads

$$\begin{aligned} \partial_{\bar{z}} T + \frac{1}{4} \partial_z \Theta &= 0, \\ \partial_z \overline{T} + \frac{1}{4} \partial_{\bar{z}} \Theta &= 0. \end{aligned} \tag{1.9}$$

At a critical point, $\Theta = 0$, so that T depends only on z, and $\overline{T}$ only on $\bar{z}$.

The stress tensor may be viewed as the generator of scale and conformal transformations, in the following way. Consider a transformation $r^\mu \to r^\mu + \alpha^\mu(r)$ which reduces to a combined scale transformation and rotation, with $\alpha(z) = \alpha z$, inside the disc $|r| < \epsilon_1$, and for which $\alpha^\mu = 0$ for $|r| > \epsilon_2$, with $\epsilon_2 > \epsilon_1$. This latter condition ensures that the system is not disturbed at infinity. In the region $\epsilon_1 \leq |r| \leq \epsilon_2$, $\alpha^\mu(r)$ may be taken to be any differentiable function.

Figure 1.2 The transformation $r^\mu \to r^\mu + \alpha^\mu$ corresponds to a scale transformation and rotation for $|r| < \epsilon_1$, and reduces to the identity for $|r| > \epsilon_2$.

If we imagine performing some generalized renormalization group transformation corresponding to this rescaling, the action will change in the region $\epsilon_1 \le |r| \le \epsilon_2$ by an amount given by (1.7). On integrating by parts, we find a bulk contribution proportional to the integral of $\alpha^\mu \partial^\nu T_{\mu\nu}$ over the annulus, which must vanish because α^μ is arbitrary there – hence $\partial^\nu T_{\mu\nu} = 0$ – and two surface contributions. That from $|r| = \epsilon_2$ vanishes because $\alpha^\mu = 0$ there, and we are left with

$$\delta S = -\frac{1}{2\pi} \int_{|r|=\epsilon_1} T_{\mu\nu} \alpha^\mu dS^\nu = \frac{\alpha}{2\pi i} \int_{|r|=\epsilon_1} T(z) z dz - \frac{\bar{\alpha}}{2\pi i} \int_{|r|=\epsilon_1} \overline{T}(\bar{z}) \bar{z} d\bar{z}. \tag{1.10}$$

On the other hand, if we consider how a correlation function $\langle \phi(0,0) \ldots \rangle$ of a scaling operator at $r = 0$ with others outside $|r| = \epsilon_2$ behaves under such a transformation, we have

$$\langle \phi(0,0) \ldots \rangle_S = (1+\alpha)^h (1+\bar{\alpha})^{\bar{h}} \langle \phi(0,0) \ldots \rangle_{S+\delta S}. \tag{1.11}$$

Expanding the term δS out of the action, and comparing coefficients of α and $\bar{\alpha}$,

$$\frac{1}{2\pi i} \int z T(z) \phi(0,0) dz = h \phi(0,0), \tag{1.12}$$

together with a similar equation involving $\overline{T}$ and $\bar{h}$. Thus the operator product expansion of $T(z)$ with ϕ has the form

$$T(z)\phi(0,0) = \cdots + \frac{h}{z^2} \phi(0,0) + \frac{1}{z} \partial_z \phi(0,0) + \cdots, \tag{1.13}$$

where the z^{-1} term comes from a similar argument to the above using a translation rather than a scale transformation. The terms exhibited in (1.13) occur in the operator product expansion of T with any scaling operator. As shown by BPZ, and discussed elsewhere in these lectures, there exist *primary* operators for which the displayed terms in (1.13) are the most singular ones. For these operators, we may consider a general transformation α^μ which is conformal for $|r| < \epsilon_1$, that is $\alpha^z = \alpha(z)$, and once again reduces to the identity for $|r| > \epsilon_2$. Running the above argument backwards, we then find that a primary operator transforms under an infinitesimal conformal transformation according to

$$\phi(z, \bar{z}) \to \left(1 + h \alpha'(z) + \bar{h} \overline{\alpha'(z)}\, \right) \phi(z', \bar{z}'). \tag{1.14}$$

For a *finite* conformal transformation $z \to z' = f(z)$, the integrated version of (1.14) is

$$\phi(z, \bar{z}) \to (f'(z))^h (\overline{f'(z)})^{\bar{h}} \phi(z', \bar{z}'). \tag{1.15}$$

The stress tensor $T(z)$ is a scaling operator with scaling dimensions $(2,0)$. Unlike other operators, we cannot arbitrarily normalize it as in (1.3) because its normalization is already fixed by its definition. Its 2-point function therefore has the form

$$\langle T(z)T(0)\rangle = \frac{c/2}{z^4}. \tag{1.16}$$

We take this as the definition of the *conformal anomaly number* or *central charge* c. It is a universal number for the the particular fixed point theory we are discussing. Eq. (1.16) implies that $T(z)$ is not a primary operator. The singular terms in the operator product expansion of T with itself are

$$T(z)T(0) = \frac{c/2}{z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}T'(0) + \cdots. \tag{1.17}$$

The first term implies the presence of an anomaly in the transformation law for T under an infinitesimal transformation. Generalizing the arguments above, we find that

$$T(z) \to (1 + 2\alpha(z))T(z') + \frac{c}{12}\alpha'''(z). \tag{1.18}$$

We shall need the integrated form of this for a finite conformal transformation $z \to z' = f(z)$. This is given by

$$T(z) \to (f'(z))^2 T(z') + \frac{c}{12}\{z', z\}, \tag{1.19}$$

where $\{z', z\}$ denotes the Schwartzian derivative

$$\{z', z\} = \frac{f''' f' - \frac{3}{2} f''^2}{f'^2}. \tag{1.20}$$

This result is very useful for calculating $\langle T(z')\rangle$ given that we know $\langle T(z)\rangle$. In the full plane or the upper half plane, for example, the latter quantity vanishes by translational invariance.

Although it is quite easy to verify that (1.18) is the infinitesimal form of (1.19), the reverse step is not so obvious. It is easier to understand the origin of the appearance of the Schwartzian derivative within a particular model. Since we shall be using this model a great deal in what follows, let us describe this calculation in some detail.

1.4 The Gaussian model

This is described, in the continuum limit, by the action for a free scalar field $\phi(r)$

$$S = \frac{g}{4\pi} \int (\partial^\mu \phi)(\partial_\mu \phi) d^2 r, \tag{1.21}$$

where g is a dimensionless coupling constant. This model is critical for all values of g. The Green's function $G(z, \bar{z}) \equiv \langle \phi(z, \bar{z}) \phi(0,0) \rangle$ is given by

$$G(z, \bar{z}) - G(0,0) = -(1/2g) \ln z - (1/2g) \ln \bar{z} + \text{const.} \tag{1.22}$$

Note that ϕ is not a scaling operator, but $\partial\phi \equiv \partial_z \phi$ and $\overline{\partial}\phi \equiv \partial_{\bar{z}}\phi$ are. Classically, we may find the stress tensor by letting $r^\mu \to r^\mu + \alpha^\mu(r)$ and comparing with (1.7). We find

$$T = -g(\partial\phi)^2, \qquad \overline{T} = -g(\overline{\partial}\phi)^2, \tag{1.23}$$

with $\Theta = 0$, as expected. However, the expressions above contain singular terms as the cutoff $a \to 0$, reflecting the infinite zero-point energy of the theory. The easiest way to regularize them is by point-splitting: the expressions above are replaced by the limits

$$:(\partial\phi)^2 :\equiv \lim_{\delta \to 0} \left[\partial\phi(z + \frac{1}{2}\delta) \partial\phi(z - \frac{1}{2}\delta) - \frac{1}{2g\delta^2} \right], \tag{1.24}$$

together with a similar definition for $\overline{T}$. Let us now calculate c for this theory. This is simple using Wick's theorem. The subtractions in (1.24) mean that we ignore terms where contractions occur between the points which become identical as $\delta \to 0$. The surviving contractions are shown in Fig. 1.3.

Figure 1.3 Wick contractions involved in calculating $\langle TT \rangle$ in the Gaussian model.

We obtain

$$\langle T(z_1)T(z_2)\rangle = g^2 \cdot 2 \cdot \left(\frac{1}{2g}\right)^2 \frac{1}{(z_1-z_2)^4}, \tag{1.25}$$

from which we read off $c = 1$, independent of g.

Let us now see how this T transforms under a finite conformal transformation $z \to z' = f(z)$. $\partial\phi$ transforms as a primary operator: $\partial\phi(z) \to f'(z)\partial\phi(f(z))$. Thus

$$\begin{aligned} T(z) &\to g \lim_{\delta\to 0}\left[f'(z+\frac{1}{2}\delta)f'(z-\frac{1}{2}\delta)\partial\phi(f(z+\frac{1}{2}\delta)\partial\phi(f(z-\frac{1}{2}\delta) - \frac{1}{2g\delta^2}\right] \\ &= \lim_{\delta\to 0}\left[f'(z+\frac{1}{2}\delta)f'(z-\frac{1}{2}\delta)\left(T(f(z)) + \frac{1/2}{(f(z+\frac{1}{2}\delta)-f(z-\frac{1}{2}\delta))^2}\right) - \frac{1}{2\delta^2}\right]. \end{aligned} \tag{1.26}$$

We see that the anomalous term is

$$\lim_{\delta\to 0}\left[\frac{f'(z+\frac{1}{2}\delta)f'(z-\frac{1}{2}\delta)}{2(f(z+\frac{1}{2}\delta)-f(z-\frac{1}{2}\delta))^2} - \frac{1}{2\delta^2}\right]. \tag{1.27}$$

The algebra is left as an exercise. The result is

$$\frac{1}{12}\frac{f'''f' - \frac{3}{2}f''^2}{f'^2} = \frac{1}{12}\{z', z\}, \tag{1.28}$$

confirming (1.18) with $c = 1$.

2. Finite-size scaling of the free energy

2.1 Finite-size scaling at criticality

The finite-size scaling hypothesis [2] has been very powerful in the interpretation of of numerical studies of critical behavior, which have perforce been carried out in finite systems. Its statement is simple. If some quantity (*e.g.* the susceptibility χ) diverges in the infinite system like ξ^{d-2x}, where ξ is the correlation length (and, in this case, x is the scaling dimension of the magnetization operator), then, in a finite system of characteristic size L, it will be given by the scaling law

$$\chi(\xi, L) = L^{d-2x}\Phi(L/\xi), \tag{2.1}$$

where Φ is some universal scaling function. At the critical point, where $\xi^{-1} = 0$, this implies that the susceptibility is proportional to L^{d-2x}. The finite-size scaling hypothesis has been put on a firm basis within the ϵ-expansion [3], and has been shown to be valid for systems below their upper critical dimensionality, when they are equivalent, in the critical region, to a renormalized field theory.

The scaling of the reduced free energy $F \equiv -\ln Z$ at criticality is less well understood. One expects an expansion for large L which, in two dimensions, begins

$$F = AL^2 + BL + \cdots \tag{2.2}$$

The coefficients A and B are respectively the bulk free energy per unit area, and the boundary free energy per unit length (in the case where the system has a boundary). Neither of these are expected to be universal (since they clearly have a dimensional dependence), and thus they are not of great theoretical interest. The corrections to (2.2) were originally supposed [4] to be $O(L^0)$, and therefore to be universal, depending only on the shape of the system, and, perhaps, the general type of boundary condition. However, we shall show that, at least for $d = 2$, there exists in general a term $O(\ln L)$ in (2.2). Its coefficient is in general highly universal, being proportional to the value of c and otherwise depending only on the topology of the manifold on which the theory is defined. A more detailed version of the arguments of this chapter appears in Ref. 5.

To understand the origin of this term, let us consider a theory defined on some manifold $\mathcal{M}$ with a given metric, with respect to which the characteristic size of the system is L. The metric may be flat or curved, and the boundaries $\partial\mathcal{M}$ may have curvature also. (We shall ignore the thorny question of how it is possible to define the continuum limit of a lattice theory in such a general curved geometry.) Under a global dilatation $r^\mu \to (1+\alpha)r^\mu$ the response of the action is, by the definition (1.7) of the stress tensor

$$\delta S = -\frac{\alpha}{2\pi}\int_{\mathcal{M}+\partial\mathcal{M}} \Theta\sqrt{g}d^2r. \tag{2.3}$$

In computing the change in the free energy, we must be careful. According to our renormalization group interpretation of such a dilatation, the total partition function must be invariant under such a rescaling. Thus, if the free energy of the system is denoted by $F(L)$, we have, to first order in α,

$$\mathrm{e}^{-F(L)} = \mathrm{e}^{-F(L+\delta L)-\langle\delta S\rangle}, \tag{2.4}$$

so that the change in the free energy is *minus* $\langle\delta S\rangle$. We thus see that

$$L\frac{\partial F}{\partial L} = \frac{1}{2\pi}\int_{\mathcal{M}+\partial\mathcal{M}} \langle\Theta\rangle\sqrt{g}d^2r, \tag{2.5}$$

so that there will exist a term in $F(L)$ proportional to $\ln L$ if the integral of $\langle\Theta\rangle$ over the whole manifold, including its boundary, is non-zero.

2.2 The trace anomaly

In Sec. (1.3) we showed that Θ, and in particular its expectation value $\langle\Theta\rangle$, vanish at a critical point as a result of scale invariance. However, this argument was valid only in the plane. In a curved geometry we would no longer expect $\langle\Theta\rangle$ to vanish, since the curvature provides a scale. In fact, if we assume that $\langle\Theta\rangle$ can depend only on the local geometry, then dimensional analysis and general co-ordinate invariance imply that

$$\langle\Theta\rangle = \lambda R, \tag{2.6}$$

where R is the scalar curvature, and λ is a constant. The left hand side of (2.6) is called the *trace anomaly* [6]. It is anomalous in the sense that it may appear to vanish when we look at the theory classically, that is we ignore the fact that it must be regulated, thus introducing a scale into the theory. A word about terminology here: the term 'trace' or 'conformal' anomaly is also used to describe a non-zero value for $\langle\Theta\rangle$ in flat space, when we are considering a theory away from a renormalization group fixed point, which, classically, looks as if it should be scale invariant. An example is a Lagrangian quantum field theory with a dimensionless coupling constant. From our point of view, which is not restricted to theories described by a Lagrangian, this kind of behavior is not particularly anomalous, but merely results from the theory not being at the fixed point.

The constant λ in (2.6) turns out to be related to the central charge c as defined by (1.15). The simplest way to see this is to consider the limit of weak curvature. We now describe this calculation. A momentum space version of this argument appears in Green, Schwarz and Witten [7].

A co-ordinate transformation of the type discussed in Sec. (1.3) may also be viewed as a change in the metric

$$\delta g_{\mu\nu} = \alpha_{\mu,\nu} + \alpha_{\nu,\mu}, \tag{2.7}$$

so that the change in the action is given by

$$\delta S = -\frac{1}{4\pi}\int T_{\mu\nu}\delta g^{\mu\nu} d^2r. \tag{2.8}$$

However, this expression has a wider range of applicability than (1.7), because it may also apply to changes in the metric corresponding to true changes in the geometry, not only to a co-ordinate change. As discussed in Sec. (2.1), within the interpretation of such a deformation which we adopt throughout these lectures, this means that

$$\mathrm{Tr}\,\exp\left(-S+(1/4\pi)\int T_{\mu\nu}\delta g^{\mu\nu}d^2r\right)\Bigg|_{\text{new geometry}}=\mathrm{Tr}\,\exp(-S)\Bigg|_{\text{old geometry}}. \tag{2.9}$$

Let us use this to calculate the response $\langle\delta T_{\mu\nu}\rangle$ in the expectation value of the stress tensor to an infinitesimal variation of the metric away from its flat space value, in some localized region:

$$\langle\delta T_{\mu\nu}(z,\bar z)\rangle=-1/4\pi\int\langle T_{\mu\nu}(z,\bar z)T_{\lambda\sigma}(z',\bar z')\rangle\delta g^{\lambda\sigma}(z',\bar z')d^2z'. \tag{2.10}$$

The non-zero components of the correlation function of the stress tensor in the plane are $\langle TT\rangle=(c/2)(z-z')^{-4}$ and $\langle\overline{T}\,\overline{T}\rangle=(c/2)(\bar z-\bar z')^{-4}$. Thus

$$\langle\delta T(z,\bar z)\rangle=-\frac{c}{8\pi}\int\frac{\delta g^{zz}(z',\bar z')}{(z-z')^4}H((z-z')(\bar z-\bar z')-a^2)d^2z', \tag{2.11}$$

together with a similar equation for $\langle\delta\overline{T}\rangle$. In writing (2.11) we have inserted a step function short-distance cutoff to render the integral finite. Because of this, $\langle T\rangle$ has explicit dependence on $\bar z$:

$$\partial_{\bar z}\langle T(z,\bar z)\rangle=-\frac{c}{8\pi}\int\frac{\delta g^{zz}(z',\bar z')}{(z-z')^3}\delta(|z-z'|^2-a^2)d^2z'. \tag{2.12}$$

Expanding δg^{zz} in powers of $z'-z$ and $\bar z'-\bar z$, the leading term as $a\to 0$ which survives the angular integration is

$$\partial_{\bar z}\langle T(z,\bar z)\rangle=\frac{c}{8\pi}\frac{1}{3!}\pi\partial_z^3 g^{zz}(z,\bar z). \tag{2.13}$$

A similar equation holds for $\partial_z\langle\overline{T}(z,\bar z)\rangle$. Now the requirement $T_{\mu\nu}{}^{;\mu}=0$ that the stress tensor be conserved implies, to first order in $\delta g_{\mu\nu}$, that

$$\begin{aligned}\partial_{\bar z}T+\partial_z T_{\bar z z}&=0,\\ \partial_{\bar z}T_{z\bar z}+\partial_z\overline{T}&=0.\end{aligned} \tag{2.14}$$

We therefore expect non-zero contributions to

$$\langle T_{\bar z z}\rangle=-\frac{c}{48}\partial_z^2 g^{zz}+\cdots \tag{2.15}$$

and

$$\langle T_{z\bar z}\rangle = -\frac{c}{48}\partial_{\bar z}^2 g^{\bar z\bar z} + \cdots. \tag{2.16}$$

We missed these contributions to (2.10) because we failed to recognize that the introduction of a cutoff explicitly breaks the conformal invariance in the plane, so that the correlation functions $\langle T_{\bar z z}T_{\mu\nu}\rangle$ are not necessarily zero, for separations of $O(a)$. In the continuum limit, these may become derivatives of delta functions, giving a the non-zero contribution to $\langle T_{\bar z z}\rangle$ shown above. There may also be contributions involving $g^{\bar z z}$ and $g^{\bar z\bar z}$. The form of these is fixed by the requirements that $T_{\bar z z} = T_{z\bar z}$ and that $\langle \Theta\rangle = 4\langle T_{\bar z z}\rangle$ should be invariant under coordinate reparametrizations $g^{\mu\nu}\to g^{\mu\nu} + \alpha^{\mu,\nu} + \alpha^{\nu,\mu}$. The result is that the complete form of Eqs. (2.2.10,2.2.11) is

$$\langle T_{\bar z z}\rangle = \langle T_{z\bar z}\rangle = -\frac{c}{48}(\partial_z^2 g^{zz} - 2\partial_{\bar z}\partial_z g^{\bar z z} + \partial_{\bar z}^2 g^{\bar z\bar z}). \tag{2.17}$$

One can now go back and check that the terms which must then be introduced in $\langle T\rangle$ and $\langle\overline{T}\rangle$ in order to satisfy (2.14) are local in $\delta g^{\lambda\sigma}$, so that the argument is consistent. We recognize the expression appearing in parentheses in Eq. (2.17) as the scalar curvature R, to first order in $\delta g^{\mu\nu}$. From this we find the coefficient in (2.6) to be

$$\lambda = -c/12. \tag{2.18}$$

2.3 Systems with a boundary

What happens if the manifold has a boundary? In that case, even if the metric is flat, the extrinsic curvature K of the boundary provides a scale, which may result in a non-zero value of $\langle\Theta\rangle$. Such a term would be concentrated in a delta-function on the boundary. Thus

$$\langle\Theta\rangle = \lambda' K\delta(x_\perp), \tag{2.19}$$

for some constant λ'. We can calculate its value by the same kind of modified perturbation theory used above. Consider now a geometry differing infinitesimally in a localized region from the upper half plane. Eq. (2.10) again applies, but we now have to use the 2-point function of the stress tensor in the upper half plane.

So far we have not discussed statistical mechanics systems with boundaries. In a spin system, we may imagine two simple kinds of boundary condition: (a) where the spins on the boundary are free, and (b) where they are all fixed to some value. Investigations of these kinds of boundary conditions using the renormalization group show that, in the continuum limit, they are equivalent to either the condition $\phi = 0$, or to $\phi \propto (x_\perp)^{-x}$, respectively, on the order parameter field [8]. The main feature of these two conditions is that they contain no length scale, and are therefore conformally invariant. We may thus hope to apply conformal invariance to systems with boundaries.

In the development of the theory in the full plane, an important step was writing the change in the action (1.7) as a contour integral, and the lack of dependence of the result on the precise contour. This is equivalent to the statement that the stress tensor is conserved. In the semi-infinite system, the contour will include part of the real axis. We are, however, required to consider only those tranformations which preserve the real axis, *i.e.*, for which $\alpha^y(x,0) = 0$. The remaining component α^x will couple to T_{xy}. For the result to be contour independent, we must then have $T_{xy} = 0$ on the real axis. In complex co-ordinates, this means that $T = \overline{T}$ (see Fig. 2.1).

Figure 2.1 Modification of the contour of Fig. 1.2 for a semi-infinite system. Part of the contour now runs along the real axis. The contribution from this vanishes if $T = \overline{T}$.

But $T(z)$ is an analytic function of z in the upper half plane. Therefore $\overline{T}(\bar{z})$ is the analytic continuation of $T(z)$ to the lower half plane. This implies that the correlation function $\langle T\overline{T}\rangle$ is no longer zero, in fact

$$\langle T(z)\overline{T}(\bar{z}')\rangle = (c/2)(z-\bar{z}')^{-4}. \tag{2.20}$$

Thus (2.11) is replaced by

$$\begin{aligned}\langle \delta T(z,\bar{z})\rangle = &-\frac{c}{8\pi}\int \frac{\delta g^{zz}(z',\bar{z}')}{(z-z')^4}H((z-z')(\bar{z}-\bar{z}'))d^2z' \\ &-\frac{c}{8\pi}\int \frac{\delta g^{\bar{z}\bar{z}}(z',\bar{z}')}{(z-\bar{z}')^4}H((z-\bar{z}')(\bar{z}-z'))d^2z',\end{aligned} \tag{2.21}$$

where now the integrals run over only the upper half plane. It is simplest to rewrite this as a single integral over the whole plane, which has the form of (2.11) with δg^{zz} replaced by h^{zz}, where

$$h^{zz}(x,y) = g^{zz}(x,y)H(y) + g^{\bar{z}\bar{z}}(x,-y)H(-y), \tag{2.22}$$

and $H(y)$ is the step function, equal to 1 for $y>0$, and zero for $y<0$. Repeating the steps leading to (2.15), we see that $\langle T_{\bar{z}z}\rangle$ is given by a similar expression, with g^{zz} replaced by h^{zz}. Note that h^{zz} has a discontinuity across $y=0$ of $g^{zz}-g^{\bar{z}\bar{z}}=4ig_{xy}(x,0)$, and its derivative with respect to y has a discontinuity of $2(g_{xx,y}(x,0)-g_{yy,y}(x,0))$. Introducing the antisymmetric step function $\mathrm{sign}(y)\equiv 2H(y)-1$, h^{zz} may then be decomposed as

$$h^{zz}(x,y) = 2ig_{xy}(x,0)\,\mathrm{sign}(y) + (g_{xx,y}(x,0)-g_{yy,y}(x,0))y\,\mathrm{sign}(y) + \text{regular piece}. \tag{2.23}$$

The significance of this decomposition is seen more readily in momentum space: the first two terms behave as $p_y\to\infty$ as p_y^{-1} and p_y^{-2} respectively, while the regular piece is $o(p_y^{-2})$. The contribution to the Fourier transform of $\langle T_{\bar{z}z}\rangle$ is then found by multiplying by $p_z^2=\frac{1}{4}(p_x^2-p_y^2-2ip_xp_y)$. A $\delta(y)$ term in $\langle T_{\bar{z}z}\rangle$ corresponds to a constant term surviving in the limit $p_y\to\infty$ in its Fourier transform. Thus the regular piece cannot contribute to the delta function, and we may neglect it. The leading term as $p_y\to\infty$ comes from p_y^2 acting on the first term of (2.23). However, this is pure imaginary, and can be shown to be cancelled when the result is added to $\langle T_{z\bar{z}}\rangle$. The finite pieces correspond, in real space, to the terms

$$\frac{1}{4}(-2i\partial_x\partial_y)2ig_{xy}\,\mathrm{sign}(y) - \frac{1}{4}\partial_y^2(g_{xx,y}-g_{yy,y})y\,\mathrm{sign}(y). \tag{2.24}$$

Recalling that the components of the metric in the above expression contain no y-dependence, we then find a delta function contribution to $\langle\Theta\rangle = 2\langle T_{\bar{z}z}\rangle + 2\langle T_{z\bar{z}}\rangle$ of

$$-\frac{c}{6}\delta(y)\big(g_{xy,x}-\frac{1}{4}(g_{xx,y}-g_{yy,y})\big). \tag{2.25}$$

However, as in the bulk calculation, this is not the whole story. There may be terms in $\langle\Theta\rangle$ depending on $g^{\bar{z}\bar{z}} = g_{xx} + g_{yy}$ which are not yet included. These contributions are fixed by the requirement that the delta function term in $\langle\Theta\rangle$ should depend only on the extrinsic curvature and be insensitive to coordinate changes which leave this qauntity invariant. Now the extrinsic curvature is given by

$$K = g_{xy,x}(x,0) - \frac{1}{2} g_{xx,y}(x,0), \tag{2.26}$$

to lowest order in $g_{\mu\nu}$. A simple way to see this is to consider the coordinate transformation

$$\begin{aligned} x' &= x + a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + \cdots \\ y' &= y + b_{11}x^2 + 2b_{12}xy + b_{22}y^2 + \cdots \end{aligned} \tag{2.27}$$

which maps the line $y = 0$ into $y' = b_{11}x'^2 + \cdots$, so that $K = 2b_{11}$. On evaluating the derivatives of $g_{ij} = \delta_{ij} + x'_{i,j} + x'_{j,i}$, it is then straightforward to verify that the right hand side of (2.26) does indeed give the correct result.

Therefore we must add a term to (2.25) which depends only on $g_{xx} + g_{yy}$ and its derivatives, so that the result only depends on the combination in (2.26). Clearly this is possible, by adding $\frac{1}{4}(g_{xx,y} + g_{yy,y})$ to the second factor. We then obtain the result (2.19), evaluated to first order in the perturbation of the metric, with $\lambda' = -c/6$.

Combining the bulk and surface contributions, the integrated trace anomaly is

$$\int_{\mathcal{M}+\partial\mathcal{M}} \Theta\sqrt{g}d^2r = -\frac{c}{12}\left[\int_{\mathcal{M}} R\sqrt{g}d^2r + 2\int_{\partial\mathcal{M}} K dS\right]. \tag{2.28}$$

The expression in square brackets in this equation is well known in differential geometry: it is given by the Gauss-Bonnet theorem to be equal to $4\pi\chi$, where χ is the Euler number, a topological invariant of the manifold. It is equal to $2 - 2h - b$, where h is the number of handles, and b is the number of boundaries. Putting together all these results (2.1.5,2.3.10) we then find the simple result for the logarithmic term in the free energy

$$F \sim -\frac{c\chi}{6}\ln L. \tag{2.29}$$

This is only the first example of the ubiquity of the number c.

2.4 Corners on the boundary

The above analysis assumed that there are no singularities, either in the metric or on the boundary. Let us consider now the case of a corner on the boundary, with internal angle γ. Assuming once again that the contribution to $\langle\Theta\rangle$ may be expressed in terms of the local geometry, we are free to choose a simple geometry in which to calculate the possible delta-function contribution from the corner. The simplest is an infinite wedge, related to the upper half plane by the conformal mapping $w = z^{\gamma/\pi}$. Calculating the Schwartzian derivative (1.3.12, 1.3.13) and using the fact that $\langle T(z)\rangle = 0$ in the upper half plane, we find that the expectation value of the stress tensor in the wedge geometry is

$$\langle T(w)\rangle = \frac{c}{24w^2}\left(1-(\pi/\gamma)^2\right), \tag{2.30}$$

together with a similar expression for $\langle\overline{T}(\bar{w})\rangle$. The fact that $\langle T\rangle$ behaves like $1/w^2$ as $w \to 0$ implies that $\langle\Theta\rangle$ has a delta-function singularity at $w = 0$. To see this, consider the integral of $\langle\Theta\rangle$ over a small neighborhood of the corner:

$$\int_{\text{corner}} \langle\Theta\rangle d^2w = \int_{\text{corner}} \frac{\partial w^\mu}{\partial w_\nu}\langle T_{\mu\nu}\rangle d^2w = \int w^\mu \langle T_{\mu\nu}\rangle dS^\nu, \tag{2.31}$$

where we have used the conservation of $T_{\mu\nu}$. The last integral, when evaluated in complex coordinates, becomes

$$-i\int w\langle T(w)\rangle dw + i\int \bar{w}\langle\overline{T}(\bar{w})\rangle d\bar{w}. \tag{2.32}$$

Substituting the explicit expressions for $\langle T\rangle$ and $\langle\overline{T}\rangle$, the contribution to the integrated trace anomaly from the corner is $(c\gamma/12)(1-(\pi/\gamma)^2)$. This implies that the contribution to the logarithmic term in the free energy from the corner is

$$\Delta F = \frac{c\gamma}{24\pi}\left(1-(\pi/\gamma)^2\right)\ln L. \tag{2.33}$$

It is interesting to note that this result is *not* what one might expect to obtain from (2.19) with a delta function singularity in the curvature, which would give an expression similar to the above, but with the factor in parentheses replaced by $2(1-(\pi/\gamma))$. These two expressions do, however, agree in the limit $\gamma \to \pi$, so that if we approximate a smooth boundary by a sequence of polygonal ones, we do get the correct result. Another way of putting this is to say that when there are corners on the boundary, the logarithmic terms in the free energy do not add up to give a result proportional to the Euler number.

A similar analysis may be applied to the case of a conical singularity in the bulk metric. The above calculation may simply be taken over, except that the conformal mapping is now $w = z^{\gamma/2\pi}$, from the whole z-plane. The semi-angle subtended by the cone is then $\arccos(\gamma/2\pi)$. The result for the contribution to the free energy from the conical singularity is then

$$\Delta F = \frac{c\gamma}{24\pi}\left(1-(2\pi/\gamma)^2\right)\ln L. \tag{2.34}$$

Once again, this is not the result we would obtain from (2.6) with a delta-function in R, which would be proportional to $(\gamma-2\pi)$, although they agree to first order in $(\gamma-2\pi)$.

2.5 On hearing the shape of a drum

The above results have been known for some time for the case of the Gaussian model defined in Sec. (1.4). For this model, the free energy is related to the logarithm of the determinant of the Laplacian in the geometry under consideration:

$$F = (1/2)\ln\det(-\nabla^2). \tag{2.35}$$

In a now classic paper, M. Kac [9] asked the question whether the shape of the boundary is completely determined by the spectrum of the Laplacian. As a step in this direction, he, and subsequently other authors [10], showed how the asymptotic distribution of the eigenvalues λ_n is related to the shape of the boundary. Explicitly, one considers the heat kernel

$$D(t) \equiv \mathrm{Tr}\,\mathrm{e}^{-t(-\nabla^2)}. \tag{2.36}$$

The asymptotic behavior of the λ_n as $n\to\infty$ is related to the small t behavior of $D(t)$. Formally,

$$\begin{aligned} 2F = \sum_n \ln\lambda_n &= -\frac{\partial}{\partial s}\sum_n \lambda_n^{-s}\Big|_{s=0} \\ &= -\frac{\partial}{\partial s}\left(\Gamma(s)^{-1}\int_0^\infty t^{s-1}D(t)dt\right)\Big|_{s=0} \\ &= -\int_\epsilon^\infty \frac{D(t)}{t}dt, \end{aligned} \tag{2.37}$$

where we have inserted an ultraviolet cutoff $\epsilon \sim a^2$. The methods of Kac and others then show that, for a smooth boundary and metric,

$$D(t) \sim A\frac{L^2}{t} + B\frac{L}{t^{1/2}} + \frac{1}{6}\chi + o(1). \tag{2.38}$$

The first two terms give the non-universal bulk and surface contributions. The last term gives something which diverges like $(\chi/6)\ln a$, as $a\to 0$. To be dimensionally correct, this should really be

$$F \sim -\frac{\chi}{6}\ln(L/a), \tag{2.39}$$

in agreement with (2.29) with $c=1$. The analysis of this section may thus be viewed as an independent derivation of this result.

3. Theories defined on a cylinder

3.1 Transfer matrix on the cylinder

Consider now a conformally invariant theory defined on an infinitely long cylinder, which may be thought of as the strip $-\infty < u < \infty$, $0 \le v \le \ell$, with periodic boundary conditions. This is related to the plane by the conformal mapping

$$w \equiv u + iv = \frac{\ell}{2\pi}\ln z. \tag{3.1}$$

Applying the transformation law (1.19) for T, we find that

$$T(w)_{\rm cylinder} = \left(\frac{2\pi}{\ell}\right)^2\left(z^2T(z)_{\rm plane} - \frac{c}{24}\right), \tag{3.2}$$

together with a similar equation for $\overline{T}$. As discussed in Sec. (1), the logarithm of the transfer matrix acting along the cylinder in the u-direction is the integral of $\hat{T}_{uu}$:

$$\begin{aligned}\hat{H} &= (1/2\pi)\int_0^\ell \hat{T}_{uu}(v)dv \\ &= (1/2\pi)\int_0^\ell (\hat{T}(v) + \hat{\overline{T}}(v))dv \\ &= \frac{2\pi}{\ell}(L_0 + \overline{L}_0) - \frac{\pi c}{6\ell},\end{aligned} \tag{3.3}$$

where we have introduced the Virasoro generators in the plane

$$L_0 = \frac{1}{2\pi i}\oint zT(z)dz, \qquad \overline{L}_0 = -\frac{1}{2\pi i}\oint \bar{z}\overline{T}(\bar{z})d\bar{z}. \tag{3.4}$$

Since the system defined on the cylinder also has translational invariance in the v-direction, the momentum operator $\hat{k}$ commutes with $\hat{H}$. In terms of the stress tensor,

$$\hat{k} = (1/2\pi) \int_0^\ell \hat{T}_{uv} dv = \frac{2\pi}{\ell}(L_0 - \overline{L}_0). \tag{3.5}$$

We reach the important conclusion that the eigenstates of $\hat{H}$ and $\hat{k}$ are in 1–1 correspondence with those of $L_0 + \overline{L}_0$ and $L_0 - \overline{L}_0$ respectively. Let us summarize some of the results of BPZ concerning these eigenstates:

1) Eigenstates of $(L_0, \overline{L}_0)$ are in 1–1 correspondence with the scaling operators of the theory, with the corresponding eigenvalues being the scaling dimensions $(h, \bar{h})$.
2) The ground state of $L_0 + \overline{L}_0$ corresponds to the identity operator $\mathbf{1}$, with scaling dimensions $(0, 0)$.
3) The eigenstates of L_0 (and those of $\overline{L}_0$) fall into highest weight representations of the Virasoro algebra. The highest weight state in a given representation (that with the lowest value of h) corresponds to a primary operator. The h-values of the other states differ by integers from that of the highest weight state.

Thus, there is a 1–1 correspondence between the eigenstates of $\hat{H}$, $\hat{k}$ and the scaling operators of the theory. The corresponding eigenvalues are [11]

$$E = E_0 + \frac{2\pi x}{\ell} \quad \text{and} \quad k = \frac{2\pi s}{\ell}, \tag{3.6}$$

where x and s are the scaling dimension and the spin of the operator. The ground state energy of $\hat{H}$ is [12]

$$E_0 = -\frac{\pi c}{6\ell}. \tag{3.7}$$

The above results give a powerful way of numerically (or analytically) calculating the value of c and of the scaling dimensions for a given model. This is because it is relatively easy to find the low-lying eigenvalues of $\hat{H}$ for finite ℓ. In using these formulas, it is, however, necessary to realize that they were derived for a theory which is at a renormalization group fixed point, and is therefore exactly conformally invariant. In practice, there will in general exist correction to scaling terms in (3.6) and (3.7), due to irrelevant operators.

Let us now verify (3.7) for the Gaussian model of Sec. (1.4). The Hamiltonian $\hat{H}$ of this model, is, before any regularization, that which we would derive classically

$$\hat{H} = \frac{1}{2} \int_0^\ell \left[(2\pi/g)\pi^2 + (g/2\pi)(\partial_v \phi)^2 \right] dv, \tag{3.8}$$

where π and ϕ obey canonical commutation relations. This may be diagonalized in terms of Fourier modes of ϕ:

$$\hat{H} = \sum_k \omega_k (a_k^\dagger a_k + \frac{1}{2}), \tag{3.9}$$

so that the ground state energy is just $\frac{1}{2}\sum_k \omega_k$. The allowed values of k in this sum are integer multiples of $2\pi/\ell$.

Suppose that we regularize the theory by putting it on a lattice of unit spacing. Then we have a lattice dispersion relation $\omega_k = (2 - 2\cos k)^{1/2}$, and $|k| \leq \pi$. The ground state energy is then

$$E_0 = \frac{1}{2}\omega(0) + \sum_{n=1}^{\ell/2-1} \omega(2\pi n/\ell) + \frac{1}{2}\omega(\pi). \tag{3.10}$$

Notice that the $k = \pi$ mode gets counted only once. Now apply the Euler-Maclaurin summation formula

$$\frac{1}{2}f(0) + \sum_{n=1}^{N-1} f(n) + \frac{1}{2}f(N) = \int_0^N f(n)dn + \frac{1}{12}(f'(N) - f'(0)) + \cdots \tag{3.11}$$

with $N = \ell/2$. (ℓ is assumed even; the whole calculation can be repeated when it is odd, to obtain the same final result.) The first term on the right hand side gives a bulk contribution proportional to ℓ in E_0. It does not appear in (3.7) because we implicitly subtracted it when we asserted that $\langle T \rangle = 0$ in the plane. The term $f'(N)$ vanishes due to the symmetry of the dispersion relation, and we are left with

$$E_0 = -\frac{2\pi}{\ell} \cdot \frac{1}{12} + o(\ell^{-1}), \tag{3.12}$$

thus verifying (3.7) with $c = 1$. Note that the final result did not depend in detail on the dispersion relation, thus confirming the universality of this coefficient.

3.2 Lattice models on the cylinder

Now we are going to spend some time discussing some lattice statistical mechanics models which can be more or less exactly solved. The aim will be to calculate the ground state energy E_0 on the cylinder and hence the value of c, using (3.7) [12,13,14]. The models we shall consider are generalizations of the Ising model called the Q-state Potts models.

Imagine a square lattice with a variable s_r on the site r, which can take Q possible values. The action for this model is

$$S = -\beta \sum_{\text{bonds}} \delta_{s_r s_{r'}}, \tag{3.13}$$

where the sum is over nearest neighbor pairs on the square lattice. This model has a $(Q-1)$-dimensional order parameter $\mathbf{M} = (M_1, \ldots M_Q)$ with $\sum_s M_s = 0$, where

$$M_s \equiv \langle \delta_{s_r s} - Q^{-1} \rangle. \tag{3.14}$$

There is a phase transition at some β_c, such that $\mathbf{M} \neq 0$ for $\beta > \beta_c$. The partition function can be written as

$$Z = \mathrm{Tr} \prod_{\text{bonds}} \left\{ 1 + (\mathrm{e}^\beta - 1)\delta_{s_r s_{r'}} \right\}. \tag{3.15}$$

If we imagine expanding this in powers of $(\mathrm{e}^\beta - 1)$, each term may be associated with a configuration of bonds on the lattice, where a bond is present (absent) depending on whether we choose the term $(\mathrm{e}^\beta - 1)$ (or 1). The bonds will form clusters, within which all the spins are in the same state, and after we perform the trace we get a factor Q for each cluster (including isolated sites). Thus

$$Z = \sum_{\text{graphs}} (\mathrm{e}^\beta - 1)^{N_B} Q^{N_C}, \tag{3.16}$$

where N_B and N_C are the numbers of bonds and clusters respectively.

To the original square lattice we now associate another one whose vertices lie at the centers of the bonds of the original lattice. Each configuration of bonds may be associated uniquely with with a set of closed loops on the new lattice, as illustrated in Fig. 3.1.

If N_S is the total number of sites of the original lattice, then the number of closed circuits in clusters is $N_B + N_C - N_S$. Thus the number of loops is

$$N_P = (N_B + N_C - N_S) + N_C, \tag{3.17}$$

so that

$$Z = Q^{N_S/2} \sum_{\text{graphs}} \left[(\mathrm{e}^\beta - 1) Q^{-1/2} \right]^{N_B} Q^{N_P/2}. \tag{3.18}$$

The critical point is known from duality arguments to be at the point where the expression in square brackets in (3.18) equals unity. Therefore, at criticality, the Potts model is equivalent to a gas of loops with a fugacity $Q^{1/2}$ for each loop.

Figure 3.1 Loop configuration on the surrounding lattice corresponding to a given configuration of bonds on the original lattice.

We can give each loop an orientation and sum over all possible orientations if we compensate correctly for the double counting. At each vertex, a loop either turns left or right if we follow the arrows. For a clockwise (anticlockwise) loop, the number of right turns minus the number of left turns will be $4(-4)$. Thus, if we associate a factor e^{iu} (e^{-iu}) with each right (left) turn, each loop will carry a total factor of $2\cos 4u$ when we sum over both orientations. Thus we can account for the fugacity factor $Q^{1/2}$ by incorporating the above factors $e^{\pm iu}$ with

$$Q^{1/2} = 2\cos 4u. \tag{3.19}$$

The resulting model may be written as a 6-vertex model [15]. This is a special case of the 8-vertex model, whose configurations are given by placing an arrow on each bond of a square lattice. The possible configurations at a vertex, together with one conventional notation for their Boltzmann weights, are shown in Fig. 3.2.

Figure 3.2 Vertex configurations and their weights in the 8-vertex model.

A special case of the 8-vertex model is the 6-vertex model, which has $d = 0$. The further specialization $a = b$ gives the F-model. The Potts model then corresponds to the case $a = b = 1$, $c = 2\cos 2u$ of the F-model. The vertex models may be solved by Bethe ansatz, and the value of c thus extracted exactly [16], but we shall proceed by a more heuristic argument [12,13,14]. The F-model may be rewritten as a solid-on-solid (SOS) model in the following way. Assign an integer-valued height variable n_R to each site R of the dual lattice. The difference $n_R - n_{R'}$ on neighbouring sites is $1(-1)$ depending on whether the the arrow on the bond we cross in going from R to R' is going to the left (right). When $d = 0$, such a unique correspondence is always possible. We then see that the different weights in the F-model depend on whether $|n_R - n_{R'}|$ is 0 or 2 for a next-nearest neighbor pair in the SOS model.

At large distances, we might expect that the fact that the n_R are restricted to discrete values should not be relevant, and that the resulting model should be in the universality class of the Gaussian model of Sec. (1.4). Such an argument may be backed up with explicit renormalization group calculations. Since we are free to rescale the field ϕ in (1.21) (thus redefining g), we choose it so that $\phi(R) = (\pi/2)n_R$. The only problem is that we do not then know the value of g to which the the 6-vertex model renormalizes.

To get around this, we need to use one piece of information from the exact solution. Baxter [15] tells us that if we turn on a small amount of non-zero d, the free energy has a singularity $|d|^{2/y_{8V}}$, where

$$y_{8V} = (4/\pi)\cos^{-1}(c/2a). \tag{3.20}$$

Now configurations with weight d, at which the number of arrows is not conserved, correspond in the SOS picture to points around which the field n_R has a discontinuity of ± 4, that is ϕ has a discontinuity of $\pm 2\pi$. In the Gaussian model, this is a *vortex*. In order to incorporate vortices into the model we must imagine that ϕ is an *angle* parametrizing the points of a circle, so that it is only defined mod(2π). A single vortex corresponds to the field configuration $\phi \sim \theta$, which, in a system of size L, has energy

$$\frac{g}{4\pi}\int_a^L \frac{d^2r}{r^2} \sim (g/2)\ln L. \tag{3.21}$$

From this we read off that the scaling dimension of the vortex operator is

$$2 - y_{8V} = g/2. \tag{3.22}$$

This is the missing piece of information we need. Now we can use simple properties of the Gaussian model to calculate quantities in the vertex models, and hence the Potts model. The relation between Q and the renormalized coupling constant is

$$\begin{aligned} Q &= 2 + 2\cos(\pi g/2), \qquad (2 \le g \le 4) \\ u &= (\pi/8)(2 - g/2). \end{aligned} \tag{3.23}$$

However, there is an important *caveat.* The Gaussian model, and hence the vertex models, all have $c = 1$. This is not true for the Potts model in general. To understand this, we must re-examine the above mapping in the case when the model is defined on a cylinder. For convenience, we suppose that the original square lattice on which the Potts model is defined is oriented at $45°$ to the axis of the cylinder, so that the arrows of the vertex model point along either the u or v axes.

The problem is that on the cylinder, loops can wrap all the way around the cylinder. We would like to give them a weight $Q^{1/2}$, but the difference between the number of right and left turns of such a loop is zero, so that in fact they are counted with a weight 2. This may be overcome by inserting a 'seam' of bonds into the vertex model. For bonds which cross the line $v = v_0$, we insert a factor e^{4iu}, or e^{-4iu}, depending on the direction of the arrow. Loops which go once around the cylinder will then pick up a factor $2\cos 4u$ as required, once we sum over both orientations. In terms of the SOS model, this is the same as putting in a factor $e^{4iu(n_R-n_{R'})}$ for each bond of the dual lattice which lies along the seam. These combine to give an overall factor of $e^{(8iu/\pi)(\phi(\infty,v_0)-\phi(-\infty,v_0))}$ when renormalized onto the Gaussian model.

Thus the overall effect is that of inserting operators $e^{\pm(8iu/\pi)\phi}$ at either end of the cylinder in the Gaussian model. In the plane, the 2-point function of these operators behaves like

$$\langle e^{(8iu/\pi)\phi(z,\bar z)}e^{-(8iu/\pi)\phi(0,0)}\rangle = \exp\left(-\frac{1}{2}(8u/\pi)^2\langle(\phi(z,\bar z)-\phi(0,0))^2\rangle\right)\sim z^{-2h}\bar z^{-2\bar h}, \tag{3.24}$$

where $h=\bar h=(8u/\pi)^2/4g$, using (1.22). According to (3.6) and (3.7), the lowest eigenstate of $\hat H$ which contributes when this operator is inserted at the ends of the cylinder has energy

$$-\frac{\pi}{6\ell}+\frac{2\pi}{\ell}\cdot\frac{(8u/\pi)^2}{2g}, \tag{3.25}$$

from which we see that the actual value of c for the Potts model is

$$c = 1-6\frac{(2-g/2)^2}{g}, \tag{3.26}$$

with g given by (3.23).

For integer values of Q we expect the theory to be unitary. We find $c=1,\frac{4}{5},\frac{1}{2},0$ for $Q=4,3,2,1$ respectively. For $Q>4$, the model has a first order phase transition and the above mappings fail. Note that for $Q=1$ there are no degrees of freedom in the theory, and $c=0$ as expected.

The above arguments represent only the simplest case of a very general relation between generalized Gaussian models with seams and vertex models. In this way, vertex models corresponding to all the values of c in the GKO construction have recently been constructed [17] (see the talks of Zuber.)

Once we know the mapping of the Potts models onto the Gaussian model, we know in principle all the critical exponents, and may even start to construct correlation functions. The problem is that the Gaussian model has an infinite number of scaling operators $e^{i\alpha\phi}$, and we do not know which ones will actually appear in the Potts model. The techniques of conformal field theory, together with the requirement of modular invariance, to which we shall turn in Sec. (4), give us this information.

3.3 Other boundary conditions

Suppose that instead of a cylinder we have a strip, of width ℓ, with boundaries. The boundary conditions are assumed to be of the conformally invariant type discussed in Sec. (2), and we also assume the same boundary condition on $v = 0$ and $v = \ell$. This geometry is related to that of the upper half plane by the conformal mapping

$$w \equiv u + iv = \frac{\ell}{\pi} \ln z. \tag{3.27}$$

Following through the same steps as in Sec. (3.1), the logarithm of the transfer matrix acting along the strip is

$$\hat{H} = \frac{\pi}{\ell} L_0 - \frac{\pi c}{24\ell}, \tag{3.28}$$

where

$$L_0 = \frac{1}{2\pi i} \int_{\text{semicircle}} zT(z)dz - \frac{1}{2\pi i} \int_{\text{semicircle}} \bar{z}\overline{T}(\bar{z})d\bar{z} = \frac{1}{2\pi i} \oint zT(z)dz = \overline{L}_0. \tag{3.29}$$

In the last two equalities, we have used the fact that $\overline{T}$ is the analytic continuation of T into the lower half plane.

Eq. (3.28) shows that the ground state energy differs by a factor of 4 from that in the periodic case. This is easy to understand in the Gaussian model. If we consider a cylinder of width 2ℓ, with periodic boundary conditions, half the modes may be taken to have nodes at $v = 0$ and $v = \ell$, and will therefore be modes satisfying Dirichlet boundary conditions for a strip of width ℓ. Thus the ground state energy of this strip is from (3.7)

$$E_0 = -\frac{1}{2} \frac{\pi}{6(2\ell)}, \tag{3.30}$$

confirming (3.28) with $c = 1$.

What are the eigenvalues of L_0 for a semi-infinite geometry? If we look at Fig. 2.1 and imagine a scaling operator $\phi(0)$ at the origin (which is on the boundary), then if $\phi(0) \to b^{-x_s}\phi(0)$ under a scale transformation, x_s will be an eigenvalue of L_0, in just the same way that in the bulk, the scaling dimension x is an eigenvalue of $L_0 + \overline{L}_0$. The quantity x_s is called the *surface* scaling dimension of ϕ. In general this is *not* equal to its bulk scaling dimension. For example, in the Gaussian model of Sec. (1.4), the bulk 2-point function of $\mathrm{e}^{ie\phi}$ is

$$\begin{aligned}\langle \mathrm{e}^{ie\phi(z_1,\bar{z}_1)} \mathrm{e}^{-ie\phi(z_2,\bar{z}_2)} \rangle &= \exp\left(-\frac{1}{2} \langle (\phi(z_1,\bar{z}_1) - \phi(z_2,\bar{z}_2))^2 \rangle \right) \\ &= \exp\left(\mathrm{e}^2 (G(z_1 - z_2, \bar{z}_1 - \bar{z}_2) - G(0,0)) \right) \sim |z_1 - z_2|^{-2x},\end{aligned} \tag{3.31}$$

with $x = e^2/2g$. In the upper half plane, the Green's function (1.22) is replaced by

$$G(z_1, \bar{z}_1 | z_2, \bar{z}_2) \sim -(1/2g)(\ln(z_1 - z_2) + \ln(z_1 - \bar{z}_2) + \ln(\bar{z}_1 - \bar{z}_2) + \ln(\bar{z}_1 - z_2)), \quad (3.32)$$

where we have taken Neumann boundary conditions $\partial_y \phi = 0$. If we take z_1 and z_2 to lie on the boundary, we then see that the 2-point function falls off as $|z_1 - z_2|^{-2x_s}$, with $x_s = 2x$. The same result is valid for Dirichlet boundary conditions, but one must then take the points slightly off the real axis. It should be emphasised that for a more general model, x_s is not simply twice the bulk scaling dimension.

To summarize, for the case of a strip with boundaries the eigenvalues of $\hat{H}$ are

$$E = E_0 + \frac{\pi x_s}{\ell}, \quad (3.33)$$

where the surface scaling dimensions x_s are eigenvalues of L_0.

4. Modular invariance on the torus

4.1 Theories defined on a torus

A torus may be thought of as a parallelogram whose opposite edges are identified. For convenience we shall consider a parallelogram whose vertices lie at $(0, 2\pi, 2\pi\tau, 2\pi(1+\tau))$. The complex number τ, with $\mathrm{Im}\tau > 0$, describes the shape of the torus, and is called the modular parameter. We normalize the free energy so that at criticality the bulk contribution is zero. There are no boundaries, and the Euler number of the torus is zero, so that, from Sec. (2), if we normalize the bulk critical free energy to vanish, the leading term in the free energy will be independent of the size, and depend only on the shape parameter τ. Thus we consider the partition function $Z(\tau, \bar{\tau})$.

The shape of the torus does not uniquely determine τ (Fig. 4.1). Because of the periodic boundary conditions, $Z(\tau, \bar{\tau})$ is invariant under

$$T : \tau \to \tau + 1. \quad (4.1)$$

If we look at the torus from the side, rather than from underneath, this also cannot change Z. This is equivalent to invariance under

$$S : \tau \to -1/\tau. \quad (4.2)$$

Figure 4.1 The torus, and different ways of looking at it corresponding to the elements S and T of the modular group.

The tranformations S and T, composed in all possible orders, generate the *modular group* $\mathrm{SL}(2,\mathbf{Z})$ of tranformations of the form

$$\tau \to \frac{a\tau+b}{c\tau+d} \qquad (a,b,c,d \in \mathbf{Z},\ ad-bc=1). \tag{4.3}$$

The generators satisfy the defining relations

$$S^2 = 1, \qquad (ST)^3 = 1. \tag{4.4}$$

It is sometimes useful, when the theory in question has some internal symmetry, to consider a generalization with twisted boundary conditions around one or both directions on the torus. In that case, Z is usually invariant under only a subgroup of the modular group.

The modular invariance of Z seems at first sight trivial. It only becomes useful when we realize that Z may be calculated in terms of the eigenvalues of $\hat{H}$, $\hat{k}$ of the theory defined on a *cylinder*. As we showed in the previous section, these are simply related to the scaling dimensions of the scaling operators in the plane. We thus get useful constraints on the latter by imposing modular invariance on the torus.

The torus may be constructed from a finite cylinder of length $2\pi\mathrm{Im}\tau$ by joining the ends, after performing a twist around the axis by $2\pi\mathrm{Re}\tau$. The operators $\hat{H}$ and $\hat{k}$ act as infinitesimal translation operators along and around the axis of the cylinder, respectively. Thus

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\,\mathrm{e}^{-2\pi(\mathrm{Im}\tau)\hat{H}+2\pi i(\mathrm{Re}\tau)\hat{k}}. \tag{4.5}$$

Using (3.3) and (3.5), this may be written

$$\begin{aligned} Z(\tau,\bar{\tau}) &= \mathrm{e}^{\pi c\mathrm{Im}\tau/6}\,\mathrm{e}^{2\pi i\tau L_0-2\pi i\bar{\tau}\overline{L}_0} \\ &= q^{-c/24}\,\bar{q}^{-c/24}\,\mathrm{Tr}\,q^{L_0}\bar{q}^{\overline{L}_0}, \end{aligned} \tag{4.6}$$

where we have introduced the notation $q \equiv \mathrm{e}^{2\pi i\tau}$. Henceforth we shall write $Z(q,\bar{q})$ as a function of q and $\bar{q}$.

4.2 The Gaussian model

Let us compute $Z(q,\bar{q})$ for the Gaussian model of Sec. (1.4). We take the field ϕ to be uncompactified, that is $-\infty<\phi<\infty$. As discussed in (3.1), we can diagonalize $\hat{H}$ and $\hat{k}$ in terms of the eigenstates

$$\prod_{k=-\infty}^{\infty} a_k^{\dagger\,n_k}|0\rangle, \tag{4.7}$$

where (since we take $\ell=2\pi$) the k are integers. The energy and momentum of this state are $-\frac{1}{12}+\sum_k|k|n_k$ and $\sum_k kn_k$ respectively. (We have taken the continuum limit so that $\omega_k=|k|$, and have already used the ground state energy computed in Sec. (3.1).) The contribution of the modes with $k\neq 0$ is

$$\begin{aligned} Z_{\{k\neq 0\}}(q,\bar{q}) &= (q\bar{q})^{-1/24}\prod_{k\neq 0}\sum_{n_k} q^{\frac{1}{2}(|k|+k)n_k}\bar{q}^{\frac{1}{2}(|k|-k)n_k} \\ &= (q\bar{q})^{-1/24}\left(\prod_{k=1}^{\infty}\sum_{n_k} q^{kn_k}\right)\left(\prod_{k=-\infty}^{-1}\sum_{n_k}\bar{q}^{-kn_k}\right) \\ &= \eta(q)^{-1}\eta(\bar{q})^{-1}, \end{aligned} \tag{4.8}$$

where we have introduced the Dedekind η-function

$$\eta(q) \equiv q^{1/24} \prod_{k=1}^{\infty} (1 - q^k). \tag{4.9}$$

The treatment of the zero mode is more subtle. When the theory is regularized properly, one finds a factor proportional to the inverse square root of the volume.

$$Z(q, \bar{q}) = \frac{g^{1/2}}{(\mathrm{Im}\tau)^{1/2} \eta(q) \eta(\bar{q})}. \tag{4.10}$$

It is by no means obvious that Z as given by this formula is invariant under S, although it is true by construction. It follows from the property of the Dedekind function [18]

$$\eta(\tilde{q}) = (-i\tau)^{1/2} \eta(q), \tag{4.11}$$

where $\tilde{q} = \mathrm{e}^{-2\pi i/\tau}$.

The infinite product in (4.9) has the important property that it is the inverse of the generating function for the number $P(n)$ of partitions of the integer n into positive integers.

$$\begin{aligned} \prod_{k=1}^{\infty} (1 - q^k)^{-1} &= (1 + q + q^2 + \cdots)(1 + q^2 + q^4 + \cdots)(1 + q^3 + q^6 + \cdots) \ldots \\ &= \sum_{n=0}^{\infty} P(n) q^n. \end{aligned} \tag{4.12}$$

4.3 Decomposition into characters

Now return to (4.6). As discussed earlier, the eigenstates of $(L_0, \overline{L}_0)$ are organized into highest weight representations of the direct sum of two Virasoro algebras. Thus we can decompose the sum implied by the Tr operation into a sum over these representations:

$$Z(q, \bar{q}) = \sum_{h, \bar{h}} N_{h\bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q}). \tag{4.13}$$

We have labelled the highest weight representations by the eigenvalue h of L_0 in the highest weight state. The non-negative integers $N_{h\bar{h}}$ tell us how many times a given representation enters, that is, how many primary scaling operators there are with scaling dimensions $(h,\bar h)$. The q-dependence is all contained in the functions

$$\chi_h(q)\equiv q^{-c/24}\,\mathrm{Tr}_h\,q^{L_0}=q^{-(c/24)+h}\sum_{n=0}^{\infty}d_h(n)q^n, \tag{4.14}$$

where $d_h(n)$ is equal to the degeneracy of states in the representation at level n. The function $\chi_h(q)$ (where we have inserted a factor $q^{-c/24}$ for convenience) is called the *character* of the representation. Its form depends only on c and h.

The simplest case is when there are no null states in the representation. Given a highest weight state $|h\rangle$, the general state at level n has the form

$$L_{-1}^{n_1}L_{-2}^{n_2}L_{-3}^{n_3}\ldots|h\rangle \tag{4.15}$$

for non-negative integers n_k, with $\sum_k kn_k=n$. If there are no null states, all these states are linearly independent, and $d_h(n)=P(n)$. For a general representation, $d_h(n)\le P(n)$, so that for q real,

$$\chi_h(q)\le q^{-c/24+h}\sum_n P(n)q^n=q^{-(c-1)/24+h}\,\eta(q)^{-1}. \tag{4.16}$$

This result can be used to place an interesting bound on c. From (4.10) we see, in this case, that as $\mathrm{Im}\tau\to 0+$

$$\chi_h(q)\le\tilde q^{-1/24}(\mathrm{Im}\tau)^{1/2}, \tag{4.17}$$

so that

$$Z(q,q)\le\tilde q^{-1/12}\mathrm{Im}\tau\sum_{h,\bar h}N_{h\bar h}. \tag{4.18}$$

But modular invariance implies that Z must behave like $\tilde q^{-c/12}$ in this limit. We therefore see that if $c\ge 1$, the number of primary operators is necessarily infinite. A word of warning here. We have defined 'primary' with respect to the Virasoro algebra. If there is a larger infinite-dimensional algebra present (*e.g.* supersymmetry, Kac-Moody, *etc.*) it is conventional to redefine the meaning of 'primary' to be with respect to this larger algebra. Then one can show that theories with a finite number of primary operators exist only for c less than some critical value, which depends on the algebra.

4.4 Theories with $c < 1$

Let us now consider only those theories with a finite number of operators which are primary with respect to the Virasoro algebra, and which therefore have $c < 1$. For the sake of simplicity, we consider only unitary models, although several statistical mechanics models (*e.g.* the Q-state Potts model for non-integer Q) do not fall into this category. Then the work of Friedan, Qiu and Shenker (FQS) [19] tells us that

1) The allowed values of c are

$$c = 1 - \frac{6}{m(m+1)} \qquad (m = 3, 4, \ldots); \tag{4.19}$$

2) The allowed values of h are given by the Kac formula

$$h = h_{rs} \equiv \frac{(r(m+1) - sm)^2 - 1}{4m(m+1)} \qquad (1 \leq s \leq r \leq m-1); \tag{4.20}$$

3) The representation with the above highest weight has a null state at level rs.

Using this information, let us try to construct the character $\chi_h(q)$, which may be relabelled $\chi_{rs}(q)$. If there were no null states, it would be given by (4.16). Taking account of the first null state and all the states which could descend from it, we then get

$$\chi_{rs}(q) = q^{-(c-1)/24}\, \eta(q)^{-1}\, q^{h_{rs}}(1 - q^{rs} + \cdots), \tag{4.21}$$

Now the null state has weight

$$h_{rs} + rs = h_{r,-s} = h_{m+r,m+1-s}, \tag{4.22}$$

and therefore itself contains a null state in its family of descendents at level $(m+r)(m+1-s)$. Thus, in writing (4.21) we were overcounting, and a more correct expression is

$$\begin{aligned} \chi_{rs} &= q^{-(c-1)/24}\, \eta(q)^{-1} \left(q^{h_{rs}} - q^{h_{r,-s}}(1 - q^{(m+r)(m+1-s)} + \cdots) \right) \\ &= q^{-(c-1)/24}\, \eta(q)^{-1}\, (q^{h_{rs}} - q^{h_{r,-s}} + q^{h_{2m+r,s}} - \cdots). \end{aligned} \tag{4.23}$$

The general structure should now be fairly clear. The manner in which the null states embed is only slightly more complicated than the naive picture given above, and the final result is [20]

$$\chi_{rs}(q) = q^{-(c-1)/24}\, \eta(q)^{-1} \sum_{k=-\infty}^{\infty} (q^{h_{2mk+r,s}} - q^{h_{2mk+r,-s}}). \tag{4.24}$$

We need to consider the transformation properties of these characters under the modular group. Since the sums in (4.14) are over integer powers of q, the behavior under T is straightforward:

$$T: \quad \chi_h(q) \to e^{2\pi i(h-c/24)}\chi_h(q). \tag{4.25}$$

The sums in (4.24) are similar to those appearing in the standard elliptic theta functions, that is, they are sums over exponentials of quadratic forms in k. They share with the theta functions remarkable properties under $S: q \to \tilde{q}$. After judicious application of the Poisson sum formula, and use of (4.10), it may be shown that [21,22]

$$\chi_{r,s}(\tilde{q}) = \sum_{r',s'} S_{rs}^{r's'}\chi_{r's'}(q), \tag{4.26}$$

where

$$S_{rs}^{r's'} = \left(\frac{8}{m(m+1)}\right)^{1/2}(-1)^{(r+s)(r'+s')}\sin\frac{\pi r r'}{m}\sin\frac{\pi s s'}{m+1}. \tag{4.27}$$

From (4.25) and (4.26) we see the remarkable fact that *the χ_{rs} transform according to a finite-dimensional representation of the modular group.*

If we call this representation $\mathcal{R}_m$, it follows that bilinears like the right hand side of (4.13) transform according to $\mathcal{R}_m^* \otimes \mathcal{R}_m$. The problem of finding modular invariant combinations thus reduces to finding the identity component of this, in general reducible representation. There is an additional requirement (at least for a unitary theory) that the $N_{h\bar{h}}$ be non-negative integers, and that the identity operator appear just once, that is $N_{00} = 1$. Since the elements of the matrix $\mathbf{S}$ are in general irrational numbers, this is a very powerful constraint.

To check whether a given combination is modular invariant, we need check only its invariance under S and T. That under T follows immediately if we insist that $N_{h\bar{h}} = 0$ unless $h - \bar{h}$ is an integer. This amounts to requiring that only integer spin local operators appear in the theory. This is necessary for correlation functions to be single-valued.

Since by its construction the matrix $\mathbf{S}$ satisfies $\mathbf{S}^2 = 1$, and its elements are real, the combination with

$$N_{h\bar{h}} = \delta_{h\bar{h}} \tag{4.28}$$

is automatically invariant. This corresponds to a theory containing all the scalar primary operators allowed by the Kac formula (4.20).

With a little more ingenuity [22], it is possible to construct other invariants. First, it is helpful to consider r, s as running over the full rectangle ($1 \leq r \leq m-1, 1 \leq s \leq m$), in which each primary operator appears just twice. Consider, for example, the case when m is odd. Then it is easy to show that

$$S_{rs}^{r's'} = (-1)^{s'-1} S_{rs}^{r',m+1-s'} = (-1)^{s-1} S_{r,m+1-s}^{r's'}. \tag{4.29}$$

This implies that the space spanned by

$$\chi_{rs} + \chi_{r,m+1-s} \qquad (s \text{ odd}) \tag{4.30}$$

is an invariant subspace. Therefore if we form the diagonal combination restricted to this subspace

$$Z = \frac{1}{2} \sum_{r} \sum_{s \text{ odd}} \left| \chi_{rs} + \chi_{r,m+1-s} \right|^2, \tag{4.31}$$

the result will be invariant under S. Moreover, one may check that $h_{rs} - h_{r,m+1-s}$ is always an integer if $m = 1 (\mathrm{mod}\ 4)$, so the result is also invariant under T, and by construction the $N_{h\bar{h}}$ are non-negative integers.

It turns out that it is possible to find a similar invariant for all $m \geq 5$. The problem of finding *all* invariants with non-negative $N_{h\bar{h}}$ is much more difficult. It has been shown [23] that there are only a finite number of other possibilities, one each for $m = 11, 12, 17, 18, 29, 30$. There is a remarkable correspondence between these two infinite sequences, with a finite number of exceptional cases, and the A-D-E classification which occurs in many apparently unrelated branches of mathematics, for example the classification of the finite subgroups of SU(2), of simply-laced Lie algebras, and of the critical points of smooth functions. As yet, there is no physical explanation of why this scheme should enter into the classification of two-dimensional universality classes, although, as we shall discuss briefly in Sec. (5), there are known lattice models for each modular invariant combination.

5. Identification of operators in particular models

5.1 The fusion rules

The Kac formula (4.20) and the requirement of modular invariance furnish us with a list of all primary operators in a given theory, for each allowed value of $c < 1$. For a particular statistical mechanics model with this value of c, we still need to know the physical identification of the operators which modular invariance has told us must exist in the scaling limit. In solving this problem, the fusion rules of BPZ turn out to be very helpful. These take the form of selection rules on the operator product expansion coefficients. In the operator product expansion

$$[\phi_i] \cdot [\phi_j] = \sum_k c_{ijk} [\phi_k] \tag{5.1}$$

(where the square brackets $[\phi_k]$ imply all the descendants of the primary operator ϕ_k), the coefficient c_{ijk} is non-vanishing iff the 3-point function $\langle \phi_i \phi_j \phi_k \rangle$ is non-zero. If these operators correspond to the values (r_1, s_1), (r_2, s_2), (r_3, s_3) in the Kac formula, then a necessary condition for the non-vanishing of c_{ijk} is that

$$r_1 + r_2 + r_3 = 1 \pmod 2, \tag{5.2}$$

and that $(r_1 - 1)$, $(r_2 - 1)$, $(r_3 - 1)$ satisfy the triangle inequalities

$$(r_1 - 1) + (r_2 - 1) \geq (r_3 - 1), \quad etc. \tag{5.3}$$

Similar conditions apply with r replaced by s.

A given statistical mechanics model will usually possess certain internal symmetries which also impose selection rules on the operator product expansion coefficients. These rules must then be consistent with those given by the fusion rules. For example, (5.2) is consistent with the existence of a Z_2 symmetry under which

$$\phi_{rs} \to (-1)^{r-1} \phi_{rs}. \tag{5.4}$$

As we shall see, such a symmetry is nearly always present in a theory, and it corresponds to Kramers-Wannier duality.

5.2 The Ising model

The lowest value of c allowed in a unitary theory is $c = \frac{1}{2}$, corresponding to $m = 3$. From (3.23) and (3.26), this value of c is that of the $Q = 2$ Potts model, more commonly known as the Ising model. There is only one modular invariant combination with this value of c, the diagonal one

$$Z = |\chi_0|^2 + |\chi_{1/16}|^2 + |\chi_{1/2}|^2. \tag{5.5}$$

Apart from the identity operator $\mathbf{1}$, we have σ, with dimensions $(\frac{1}{16}, \frac{1}{16})$, and ϵ, with dimensions $(\frac{1}{2}, \frac{1}{2})$. The fusion rules give the following non-vanishing operator product expansion structure:

$$[\epsilon] \cdot [\epsilon] = [\mathbf{1}], \tag{5.6}$$

$$[\sigma] \cdot [\epsilon] = [\sigma], \tag{5.7}$$

$$[\sigma] \cdot [\sigma] = [\mathbf{1}] + [\epsilon]. \tag{5.8}$$

The action of the nearest neighbor zero-field Ising model is

$$S = -\beta \sum_{\text{bonds}} s(r)s(r'), \tag{5.9}$$

where $s(r) = \pm 1$. There is an obvious Z_2 symmetry allowed by (5.6–8)

$$\sigma \to -\sigma, \quad \epsilon \to \epsilon, \tag{5.10}$$

which suggests that we identify σ with the scaling limit of the local magnetization $s(r)$, and ϵ as the scaling limit of the local energy density $\sum_{r'} s(r)s(r')$. These identifications are of course borne out by the exact solution of the Ising model, which shows that the scaling dimensions of the magnetization and energy density are $\eta/2 = \frac{1}{8}$ and $2 - \nu^{-1} = 1$ respectively.

The symmetry under duality is more subtle. Writing the partition function as

$$Z = \mathrm{Tr} \prod_{\text{bonds}} (1 + x s(r)s(r')), \tag{5.11}$$

and expanding in powers of x as for the Potts model in Sec. (3.2), the Z_2 symmetry means that the diagrams consist of closed loops of bonds. Each diagram may be associated with a spin configuration of another Ising model on the dual lattice, in which all the spins in the interior of a loop are down, and the rest are up. Thus each bond present with a weight x corresponds to a domain wall on the dual lattice, as shown in Fig. 5.1.

If the coupling constant of the dual model is β^*, then

$$\mathrm{e}^{-2\beta^*} = x = \tanh \beta. \tag{5.12}$$

The duality symmetry implies that, apart from an unimportant constant, $Z(\beta^*) = Z(\beta)$. The critical point is at $\beta = \beta^* = \beta_c$. Near the critical point, if we define $t = \beta_c - \beta$, the

Figure 5.1 A graph in the high-temperature expansion and the corresponding dual spin configuration.

duality transformation simply reverses the sign of t. Since t couples to the energy density ϵ, we see that ϵ is odd under duality. This is consistent with the fusion rule (5.6), where a term $[\epsilon]$ would be allowed on the right hand side by Z_2 spin symmetry.

If we insert a spin $s(r)$ into the trace in (5.11), the resulting diagrams will have an odd number of bonds ending at the site r. In order to perform the duality transformation, we must therefore artificially insert a domain wall ending at r. This means that we reverse the sign of β^* on the dual bonds which cross the domain wall. The insertion of such a line of bonds corresponds [24] to the placement of a *disorder* operator μ at the end of the line (see Fig. 5.2).

The disorder operator is the dual of the spin s. It is non-local with respect to s, because $s(r)$ will change sign if r is taken in a closed loop about a disorder operator, and therefore it does not appear in the operator content implied by (5.5).

It is interesting to explore other kinds of boundary conditions on the torus consistent with the Z_2 symmetry. Suppose, for example that we impose antiperiodic boundary conditions in the $\mathrm{Im}\tau$ direction, so that we calculate

$$Z_{PA} = |\chi_0|^2 - |\chi_{1/16}|^2 + |\chi_{1/2}|^2, \tag{5.13}$$

Figure 5.2 Definition of the disorder operator. Bonds crossing the wavy line have the sign of β reversed.

in an obvious notation. Using $\mathbf{S}$ we may now calculate Z_{AP}. The result is

$$Z_{AP} = |\chi_{1/16}|^2 + \chi^*_{1/2}\chi_0 + \chi^*_0\chi_{1/2}. \tag{5.14}$$

From this we can read off the operator content on a cylinder with antiperiodic boundary conditions around the cylinder. This is conformally equivalent to the plane with a line of bonds across which the sign of β is reversed, thus placing a disorder operator at the origin, and one at infinity. Indeed, in (5.14) we see the dimensions $(\frac{1}{16}, \frac{1}{16})$ of μ, which, by duality, are the same as those of σ. The other operators are fermions ψ, $\overline{\psi}$ which are formed in the operator product expansion of σ and μ. They are related to the fermions introduced by Onsager in his original solution of the Ising model, and because they have $\bar{h} = 0$ and $h = 0$ respectively, they obey the massless Dirac equation

$$\partial_{\bar{z}}\psi = 0, \qquad \partial_z\overline{\psi} = 0. \tag{5.15}$$

Since the combination $Z_{PA} + Z_{AP}$ is, by construction, invariant under S, we see that

$$|\chi_0 + \chi_{1/2}|^2 \tag{5.16}$$

is invariant under the subgroup generated by S and T^2. (5.16) corresponds to the free fermion description of the Ising model, in which the only primary operators are $\mathbf{1}$, ψ, $\overline{\psi}$ and ϵ. The magnetization σ is non-local in this picture, and does not appear.

Finally, let us consider the operator content in the case when the strip has boundaries [25]. First, suppose that the boundary conditions are free, that is, the spins on the boundaries are unconstrained. Consider the partition function $Z_{FP}(\tau)$ of a rectangle of dimensions $(\ell/2)\times\ell\mathrm{Im}\tau$ (where τ is pure imaginary), with periodic boundary conditions in the v-direction, and with free boundary conditions on $v=0$ and $v=\ell/2$. The contribution of a particular operator of surface scaling dimension x_s to Z_{FP} is

$$\mathrm{e}^{\pi c\mathrm{Im}\tau/12}\cdot\mathrm{e}^{-2\pi x_s\mathrm{Im}\tau}, \tag{5.17}$$

using (3.28) and (3.33). If the number of primary operators with surface scaling dimension h is N_h, then the complete partition function is

$$Z_{FP}=\sum_h N_h\chi_h(q), \tag{5.18}$$

where $\chi_h(q)$ is the usual Virasoro character, and $q=\mathrm{e}^{2\pi i\tau}$. Note that in this case Z is linear, rather than bilinear, in the characters. This expression may be transformed using $\mathbf{S}$ into something linear in the $\chi_h(\tilde{q})$. This may then be interpreted as an expression for Z_{PF}, in terms of the transfer matrix in the periodic sector, whose operator content is already known to be given by (4.28).

$$Z_{PF}(\tilde{q})=\sum_{h=0,\frac{1}{16},\frac{1}{2}}|\langle F|h\rangle|^2\chi_h(\tilde{q}^{1/2})^2. \tag{5.19}$$

Note that in this case, the partition function is not a trace, because of the free boundaries represented by the state $|F\rangle$ in the above. (In fact, (5.19) is not quite correct as written, since $|F\rangle$ may couple differently to the higher states. This does not affect the argument.) Moreover, for the states corresponding to the magnetization operator σ with $h=\frac{1}{16}$, $\langle F|h\rangle$ must vanish by symmetry.

If we now equate the leading terms in the two expressions (5.18) and (5.19) as $\tilde{q}\to0$, we obtain stringent constraints from the above vanishing, and the positivity of the other terms, which completely determine the N_h. The result is [25]

$$Z_{FP}=\chi_0+\chi_{1/2}. \tag{5.20}$$

This means that there is only one primary operator in this sector, apart from the identity operator $\mathbf{1}$. This must be odd under spin reversal, and we identify it with the surface magnetization operator σ, with surface scaling dimension $x_s = 1/2$. This agrees with an exact determination of this exponent for the Ising model [26]. Note that the energy operator is no longer primary, but corresponds in fact to $L_{-2}\mathbf{1}$.

The case of fixed boundary conditions is related to that of free boundary conditions by duality. To see this, note that free conditions imply that the couplings β on bonds perpendicular to the boundary vanish. After applying a duality transformation, then, the bonds *parallel* to the boundary will have infinite β. This will freeze all boundary spins into the same state, either $s = +1$ or $s = -1$. Note, however that spins on opposite sides of the strip do not have to be equal. Thus, in an obvious notation,

$$Z_{FP} = Z_{++,P} + Z_{+-,P}. \tag{5.21}$$

In fact, it is easy to show [25] that the only primary states in the $(++)$ and $(+-)$ sectors are those with $h = 0$ and $h = \frac{1}{2}$ respectively. There is no distinction between magnetization and energy operators in either of these sectors, since the Z_2 symmetry is broken.

The above calculations illustrate a general result: modifying the boundary conditions changes the operator content. Although the scaling dimensions are still given by the Kac formula, a given physical operator may appear at different positions in the conformal grid (*i.e.* have different values of r and s), depending on the boundary conditions.

5.3 Lattice effects

It is important to realize that conformal invariance, together with modular invariance, determines not only the relevant operators, with scaling dimensions $x < 2$, but also all the irrelevant ones also. Although it is formulated in the continuum limit, it is therefore capable of classifying correction to scaling terms which will occur when the model is formulated on a lattice. The operators responsible for these will break rotational invariance down to the point group of the lattice.

As an example, consider a model on a square lattice, with equal couplings in the x- and y-directions. The rotational symmetry is broken down to D_4. The lattice action will differ from the fixed point action by quasi-primary (*i.e.* non-derivative) operators whose spin is a multiple of 4. The most relevant such operator is

$$\mathcal{O}_4 = (L_{-2}^2 - \frac{3}{5}L_{-4})\mathbf{1} + (\overline{L}_{-2}^2 - \frac{3}{5}\overline{L}_{-4})\mathbf{1}, \tag{5.22}$$

which has scaling dimension $x=4$. The leading contribution to the 2-point function of a scalar operator ϕ with scaling dimensions (h,h) is then proportional to

$$\int \langle \phi(z,\bar{z})\phi(0,0)\mathcal{O}_4(z_1,\bar{z}_1)\rangle d^2z_1 = \frac{A}{r^{4h}}\frac{\cos 4\theta}{r^2}, \tag{5.23}$$

where $z=re^{i\theta}$. The fact that the leading lattice corrections on a square lattice are $O(r^{-2})$ down on the leading term is a direct consequence of conformal invariance. This result is confirmed in detailed calculations for the spin-spin correlation function of the Ising model [27]. The way in which the amplitude A depends on h may also be determined. This simple exercise in BPZ technology is left for the student.

5.4 Landau-Ginzburg classification

The Ising model in d dimensions is usually described in field-theoretic terms as a cut-off Φ^4 Lagrangian field theory, with a Landau-Ginzburg-Wilson (LGW) action

$$S = \frac{1}{2}\int \left((\nabla\Phi)^2 + m^2\Phi^2 + \lambda\Phi^4\right) d^dr. \tag{5.24}$$

This is particularly appropriate at, and just below $d=4$, when the $\varepsilon = 4-d$ expansion may be performed. For $d=2$, this, naturally, does not give accurate numerical results. However, we would expect some of the qualitative features to persist. For example, just below four dimensions, there are two relevant operators at the fixed point, identified with Φ and Φ^2, which are respectively Z_2 odd and even (Φ^3 is redundant, *i.e.* it can be removed by a redefinition of Φ). This is what we find at $d=2$. The number of relevant operators must remain invariant as we move down from $d=4$ to $d=2$, since an irrelevant operator becoming relevant, or *vice versa*, would modify the stability of the fixed point. However, the case of irrelevant operators is different. In the LGW theory there are an infinite number, corresponding to all powers of Φ mixed with arbitrary numbers of derivatives. At $d=4$ their scaling dimensions differ by integers, but in $4-\varepsilon$ dimensions the spectrum of scaling dimensions becomes very complicated. What is almost miraculous is that in $d=2$ they again become neatly organized into primary operators and their descendants. In the Ising model, in fact, since both primary operators are relevant, *all* the correction to scaling terms which can appear due to irrelevant operators must be analytic. Of course, all of this is just a consequence of the existence of an infinite-dimensional conformal symmetry in two dimensions.

The other diagonal modular invariant combinations with $c < 1$ turn out to correspond to the universality classes of multicritical Ising models. These have only scalar operators, and correspond to LGW models

$$S = \int \left((\partial\Phi)^2 + \sum_{n=1}^{2(m-1)} \lambda_n \Phi^n \right) d^d r, \tag{5.25}$$

at the multicritical point where $\lambda_1 = \lambda_2 = \ldots = \lambda_{2m-3} = 0$. As in the case $m = 3$, the relevant operators $(\Phi, \Phi^2, \ldots, \Phi^{2m-4})$ and the leading irrelevant operator Φ^{2m-2} may be placed in 1-1 correpsondence, in order of increasing scaling dimension, with the known scaling operators at $d = 2$. The case $m = 6$ is illustrated in the table below.

5	$\frac{23}{8}$	$\frac{4}{3}$	$\frac{3}{8}$	0
$\frac{22}{7}$	$\frac{85}{86}$	$\frac{10}{21}$	$\frac{1}{56}$	Φ^4
$\frac{12}{7}$	$\frac{33}{56}$	$\frac{1}{21}$	Φ^3	Φ^8
$\frac{5}{7}$	$\frac{5}{56}$	Φ^2	Φ^7	
$\frac{1}{7}$	Φ	Φ^6		
1	Φ^5	Φ^{10}		

Note that the Z_2 symmetry $\Phi \to -\Phi$ is consistent with the fusion rules.

Lattice models corresponding to the above theories were constructed independently by Andrews, Baxter and Forrester (ABF) [28], by modifying the SOS model equivalent to the 8-vertex model in such a way as to maintain the solvability. In these RSOS models, the integer-valued varables n_R are restricted to the values $1 \leq n_R \leq m$, and the Z_2 symmetry corresponds to $n_R \to m + 1 - n_R$. ABF solved these models using the corner transfer matrix method to be described in Sec. (7).

5.5 The 3-state Potts model

The operator content of the non-diagonal modular invariant combinations is more interesting. The simplest case occurs for $m = 5$, and we know from Sec. (3.2) that this corresponds to the $Q = 3$ Potts model. The operator content in the notation $(r, s; r', s')$ corresponding to scaling dimensions $(h_{rs}, h_{r's'})$ is

$$(11;11) \quad (21;21) \quad (31;31) \quad (41;41) \tag{5.26}$$

$$(21;31) \quad (31;21) \quad (11;41) \quad (41;11) \tag{5.27}$$

$$(33;33) \quad \times 2 \quad (43;43) \quad \times 2 \tag{5.28}$$

The group Z_3 has a 2-dimensional real representation, and so it is not surprising to see doubling of the operators in (5.28). We identify (5.26) with energy-like operators (invariant under the Z_3 symmetry), and (5.27) with magnetization operators. In the LGW scheme, the simplest critical theory with a Z_3 symmetry has a complex scalar field Φ and action

$$S = \int \left((\nabla\Phi)^2 + \lambda(\Phi^3 + {\Phi^*}^3) \right) d^d r. \tag{5.29}$$

We may identify the leading relevant Z_3-invariant operator $(21;21)$ with $\Phi^*\Phi$. In addition, the leading irrelevant operator $(31;31)$ corresponds to $\Phi^3 + {\Phi^*}^3$. The leading magnetization operators $(33;33)$ are Φ and Φ^*. The equation of motion

$$-\nabla^2\Phi + 3\lambda{\Phi^*}^2 = 0, \tag{5.30}$$

together with its complex conjugate, implies that Φ^2 and ${\Phi^*}^2$ are redundant, and therefore should not appear. The other relevant magnetization operators $(43;43)$ then must correspond to ${\Phi^*}^2\Phi$ and $\Phi^*\Phi^2$. They are marginal at the upper critical dimension $d = 6$, and thus may become relevant for $d < 6$.

Finally, non-scalar primary operators are allowed in a theory with a complex scalar field. The operators $(31;21)$ and $(41;11)$, with spins 1 and 3, correspond to $(\Phi^*\partial\Phi - \Phi\partial\Phi^*)$ and $(\Phi^*\partial^3\Phi - \Phi\partial^3\Phi^*)$ respectively.

The fusion rules are consistent with the Z_3 symmetry, which, unlike the Ising case, allows $[\sigma]$ to appear in the operator product expansion $[\sigma]\cdot[\sigma]$. They are also consistent with duality: similar arguments to those in the Ising model show that the 3-state Potts model on a square lattice is self-dual. The energy operators with r even (odd) are then respectively odd (even) under duality. It is possible to play the same games as we did in the Ising model in considering the effect of different boundary conditions. The results are even richer [25].

Unfortunately, it is more difficult to construct LGW models corresponding to the non-diagonal combinations with higher values of m. The symmetry implied by the doubling of operators does not appear to be manifest. However, Pasquier [29], by observing that the state space of the ABF models could be regarded as the Dynkin diagram for the algebra A_m, and that the Boltzmann weights were simply related to eigenvectors of the Cartan matrix, was able to construct similar lattice models defined on the Dynkin diagrams of the D_n and E_n series.

6. The continuum limit away from criticality

6.1 The stress tensor

In this section we are going to assume that we know all that is to be known about conformally invariant theories, which correspond to statistical mechanics models at a critical point. What can we now say if the action is perturbed by some relevant operator? For example,

$$S = S^* - \lambda \int \phi(z,\bar{z}) d^2 z, \tag{6.1}$$

where S^* is the fixed point action, ϕ is a scalar operator with scaling dimensions (h,h), and λ is a coupling constant, with dimensions $(1-h, 1-h)$. For this to be a relevant perturbation, $h<1$, because the renormalization group eigenvalue of λ is $y = 2-2h$.

First, we have to understand clearly what is implied by the continuum notation in (6.1). It means that we are working in a renormalized quantum field theory. Thus the operator ϕ appearing in (6.1) is a renormalized operator, defined by the requirement that its correlation functions are finite in the continuum limit, and λ is a renormalized coupling constant. In general, the relation between renormalized and bare operators (*i.e.* the counterterms) will depend on the interaction, that is on λ. However, when λ is of positive dimension, there are only a finite number of additional counterterms possible. In the case when S^* corresponds to a free field theory, this is equivalent to the statement that that the theory with $\lambda \neq 0$ is *super-renormalizable.* In that case, of course, no renormalization is necessary in the $\lambda = 0$ theory. In the more general case, we can say that the renormalized operators in the $\lambda \neq 0$ theory may be expressed as a linear combination of a finite number of renormalized operators in the $\lambda = 0$ theory. This will become clearer as we consider some examples.

Suppose then we define ϕ in (6.1) to be renormalized at $\lambda = 0$. If we now calculate some connected correlation function of ϕ perturbatively in λ we find

$$\langle\phi(z,\bar z)\ldots\rangle = \langle\phi(z,\bar z)\ldots\rangle_{S^*} + \lambda\int\langle\phi(z,\bar z)\phi(z_1,\bar z_1)\ldots\rangle_{S^*}\,d^2z_1 + \cdots \tag{6.2}$$

As $z \to z_1$ the integrand in the second term may be estimated using the operator product expansion. The general term behaves like $|z-z_1|^{-2x+x_i}\phi_i$, where ϕ_i is an operator which enters the operator product expansion of ϕ with itself, and x, x_i are the scaling dimensions of ϕ and ϕ_i respectively. This singularity will be integrable if $2x - x_i < 2$. In general, there will only be a finite number of operators ϕ_i which violate this criterion. (If $x \le 1$ there will be none.)

At order λ^n, new non-integrable singularities will arise only if there exist operators coupling to ϕ^n with dimensions $x_i < x - n(2-x)$. Since by assumption $x < 2$, at some point this becomes impossible. Thus only a finite number of counterterms are required to renormalize ϕ in the $\lambda \neq 0$ theory. For convenience let us suppose that no such terms arise, either because operators ϕ_i coupling to ϕ^n do not exist, or simply because $x \le 1$.

For the stress tensor, there is always at least one counterterm. Consider the zz component T. If we calculate a correlation function involving T in perturbation theory, we find

$$\langle T(z,\bar z)\ldots\rangle = \langle T(z)\ldots\rangle_{S^*} + \lambda\int\langle T(z)\phi(z_1,\bar z_1)\ldots\rangle_{S^*}\,d^2z_1 + \cdots \tag{6.3}$$

Since we have the operator product expansion

$$\begin{aligned} T(z)\phi(z_1,\bar z_1) &= \frac{h}{(z-z_1)^2}\phi(z_1,\bar z_1) + \frac{1}{z-z_1}\partial\phi(z_1,\bar z_1) + \cdots \\ &= \frac{h}{(z-z_1)^2}\phi(z,\bar z) + \frac{1-h}{z-z_1}\partial\phi(z,\bar z) + \cdots, \end{aligned} \tag{6.4}$$

we see that the integral in (6.3) is ultraviolet divergent, and should be regulated by, for example, inserting a step function cut-off $H((z-z_1)(\bar z-\bar z_1) - a^2)$ into the integral. The most singular term vanishes on angular integration, but, as a consequence of the cutoff, $\partial_{\bar z}T$ is no longer zero. In fact

$$\partial_{\bar z}T = \lambda\int\frac{(1-h)}{(z-z_1)}(z-z_1)\partial_z\phi(z,\bar z)\delta(|z-z_1|^2 - a^2)d^2z_1 + \cdots \tag{6.5}$$

Since the stress tensor must remain conserved

$$\partial_{\bar z}T + \frac{1}{4}\partial_z\Theta = 0. \tag{6.6}$$

Comparing with (6.5) we see that

$$\Theta(z,\bar{z}) = -4\pi\lambda(1-h)\phi(z,\bar{z}) + \cdots. \tag{6.7}$$

With the assumptions we made earlier, there are no higher order terms. The coefficient in (6.7) may be simply understood if we recall that Θ is the response of the action to a scale transformation $z \to b^{-1}z$. Under this transformation in (6.1), $d^2z \to b^{-2}d^2z$, and $\phi \to b^{2h}\phi$. For $(b-1)$ infinitesimal, we then obtain the first term in (6.7). This is valid all the way along a renormalization group trajectory leaving the fixed point, until we reach another infrared stable fixed point. At any point on the trajectory the IR behaviour is determined by this second fixed point. However, precisely at the new fixed point, the ultraviolet behavior changes also. This has the effect that new renormalization of ϕ is required, and in fact $\phi_R \sim a^{-\kappa}\phi$ for some positive κ. Thus, in the renormalized theory ϕ, and therefore Θ as given by (6.7), are zero, as expected at a fixed point.

6.2 Zamolodchikov's c-theorem

Let us consider more carefully the global picture of a renormalization group flow from a UV stable fixed point, with a relevant operator ϕ, to a relatively IR stable fixed point. Both fixed points correspond to conformally invariant theories. Can we say anything about one fixed point given properties of the theory at the other? An important qualitative result in this direction is given by the c-theorem [30], which states that *there exists a function C of the coupling constants which is non-increasing along renormalization group trajectories, which is stationary only at fixed points, and which, at a fixed point, is equal to the value of c for the corresponding theory.*

The proof is based on rotational invariance, positivity, and the conservation of the stress tensor. Consider a theory at some particular point on the renormalization group trajectory specified by a set of couplings $\{g\}$. For the time being, however, we suppress the dependence on $\{g\}$. Recall that T, Θ and $\overline{T}$ are respectively the spin $2, 0, -2$ components of the stress tensor. Thus their two-point functions must have the form

$$\begin{aligned} \langle T(z,\bar{z})T(0,0)\rangle &= F(z\bar{z})/z^4, \\ \langle \Theta(z,\bar{z})T(0,0)\rangle = \langle T(z,\bar{z})\Theta(0,0)\rangle &= G(z\bar{z})/z^3\bar{z}, \\ \langle \Theta(z,\bar{z})\Theta(0,0)\rangle &= H(z\bar{z})/z^2\bar{z}^2. \end{aligned} \tag{6.8}$$

On the other hand, conservation of the stress tensor says that

$$\partial_{\bar z} T + \frac{1}{4}\partial_z \Theta = 0. \tag{6.9}$$

Taking the correlation function of this equation with $T(0,0)$ and with $\Theta(0,0)$, and using (6.8), we find two equations

$$\begin{aligned} \dot F + \frac{1}{4}(\dot G - 3G) &= 0 \\ \dot G - G + \frac{1}{4}(\dot H - 2H) &= 0, \end{aligned} \tag{6.10}$$

where $\dot F = z\bar z F'(z\bar z)$, *etc.* Eliminating G from the above and defining $C \equiv 2F - G - \frac{3}{8}H$, we see that

$$\dot C = -\frac{3}{4}H. \tag{6.11}$$

Now, by reflection positivity (remember Θ is a scalar), $H \geq 0$. Thus C is a non-increasing function of $R \equiv (z\bar z)^{1/2}$, at fixed $\{g\}$. The function $C(R,\{g\})$, being a dimensionless quantity independent of the ultraviolet cut-off, should satisfy a Callan-Symanzik equation

$$\left(R\frac{\partial}{\partial R} + \sum_i \beta_i(\{g\})\frac{\partial}{\partial g_i}\right) C(R,\{g\}) = 0, \tag{6.12}$$

where

$$\frac{dC}{dl} = -\sum_i \beta_i(\{g\})\frac{\partial C}{\partial g_i} \tag{6.13}$$

is the rate of change of C along the renormalization group trajectory, at fixed R. The above result then implies that if we define

$$C(\{g\}) \equiv C(1,\{g\}), \tag{6.14}$$

this quantity satisfies the first part of Zamolodchikov's theorem. Moreover, C is stationary iff $H = 0$ which implies $\Theta = 0$, *i.e.* that we are at a fixed point. Finally, at a fixed point, $G = H = 0$, and $F = \frac{1}{2}c$, so that indeed $C = c$.

Zamolodchikov's theorem has the interpretation that renormalization group flows go 'downhill'. In particular, it rules out the existence of limit cycles and other bizarre behavior in renormalization group flows, at least in unitary theories. Physically, one would expect such a quantity to exist, because the renormalization group coarse-graining procedure implies a loss of information about the degrees of freedom of the theory whose wavelength is of the same order as the cut-off. Thus renormalization group flows are irreversible, and there should exist some kind of entropy function which measures this loss of information. Interestingly enough, this general argument should apply not only in two dimensions, but an attempt at a naive generalization of the above derivation fails for $d > 2$, because there are more invariant amplitudes in the correlation function.

Returning to (6.11), we can use it to express the total change in C from short to large distances as

$$\Delta c = -\frac{3}{4}\int_0^\infty R^2\langle\Theta(R)\Theta(0)\rangle d(R^2) = -12\pi\lambda^2(1-h)^2\int r^2\langle\phi(r)\phi(0)\rangle d^2r. \tag{6.15}$$

This result is very interesting because it relates the change in the quantity c, a number characterizing a conformally invariant theory, to an integral of a correlation function of a physical observable measured *away* from the critical point. For example, ϕ may be the energy density ϵ, in which case λ is the temperature difference $t = \beta_c - \beta$. In the usual case, the renormalization group flows will then end at a trivial high- or low-temperature fixed point, with $c = 0$. Thus (6.15) gives a formula for c at the critical fixed point in terms of the second moment of the energy-energy correlations.

6.3 The Ising model

Let us see how this works in the free-fermion description of the Ising model. As discussed in Sec. (4), this may be formulated in terms of a pair of Majorana fermion fields $(\psi,\overline{\psi})$ satisfying the Dirac equation, so that the fixed point action is

$$S^* = \int(\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi})d^2z. \tag{6.16}$$

The energy density is $\epsilon = i\overline{\psi}\psi$ (note the i, which is required to make $\langle\epsilon\epsilon\rangle$ positive), so that moving away from the critical temperature corresponds to adding a mass term $im\int\overline{\psi}\psi d^2z$ to S, where we use m instead of t above. The propagators are

$$\begin{aligned}
\langle\overline{\psi}(z,\bar z)\psi(0,0)\rangle &= -im\int\frac{e^{\frac12 i(\bar p z+p\bar z)}}{p^2+m^2}\frac{d^2p}{(2\pi)^2} = \frac{-im}{2\pi}K_0(mR),\\
\langle\psi(z,\bar z)\psi(0,0)\rangle &= \int\frac{-i\bar p e^{\frac12 i(\bar p z+p\bar z)}}{p^2+m^2}\frac{d^2p}{(2\pi)^2} = 2\partial_{\bar z}\frac{1}{2\pi}K_0(mR) = -m\frac{(\bar z/z)^{\frac12}}{2\pi}K_1(mR),\\
\langle\overline{\psi}(\bar z,z)\overline{\psi}(\bar z,z)\rangle &= -m\frac{(z/\bar z)^{\frac12}}{2\pi}K_1(mR),
\end{aligned} \tag{6.17}$$

so that the energy-energy correlations are

$$\begin{aligned}\langle\epsilon(R)\epsilon(0)\rangle &= -\langle\overline{\psi}(R)\psi(R)\overline{\psi}(0)\psi(0)\rangle \\ &= -|\langle\overline{\psi}(R)\psi(0)\rangle|^2 + \langle\psi(R)\psi(0)\rangle\langle\overline{\psi}(R)\overline{\psi}(0)\rangle \\ &= \left(\frac{m}{2\pi}\right)^2 \left[K_1^2(mR) - K_0^2(mR)\right].\end{aligned} \tag{6.18}$$

Substituting this into (6.15), we find, after doing the integrals,

$$c = 12\pi m^2(1-\frac{1}{2})^2\left(\frac{m}{2\pi}\right)^2\cdot m^{-4}\cdot(2\pi)\cdot\frac{1}{3} = \frac{1}{2}, \tag{6.19}$$

thus confirming what we saw in Sec. (5.2).

6.4 Perturbative calculation of c

Quite commonly in renormalization group theory, if we vary some parameter like the dimensionality d or the number of components of the order parameter, it may happen that two fixed points collide at some critical value of the parameter. In that case, close to the critical value, the two fixed points will be close together, and it is possible to construct properties of the theory at one fixed point in terms of known properties of the other. The standard example of this appears in the ε-expansion, where the Gaussian and Wilson-Fisher fixed points collide for $d=4$, and one may calculate exponents, *etc.* at the latter fixed point in terms of the trivial behavior at the Gaussian fixed point. When two fixed points are close together, the renormalization group eigenvalue y along the flow connecting the fixed points is small, giving the effective expansion parameter.

In such a case, there is a simple way of constructing the 'one-loop' renormalization group equations, given the operator product expansion coefficents at the IR unstable fixed point. Although this method is by no means new, it is not as well-known as it should be. It gives, for example, the easiest and most direct way of calculating the critical exponents in the ε-expansion, to $O(\varepsilon)$. Suppose the action at the IR unstable fixed point is S^*, and this is perturbed to

$$S = S^* - \sum_i \lambda_i \int \phi_i(r)d^2r, \tag{6.20}$$

where the ϕ_i are scaling operators with scaling dimensions x_i close to 2. For the purposes of this lecture, we take $d=2$; this is not necessary. The renormalization group equations will be in terms of the dimensionless bare coupling constants $g_i \equiv a^{2-x_i}\lambda_i$. Suppose we expand the free energy in powers of the $\{g_i\}$. The general term in the expansion has the form

$$\frac{g_1^{n_1} g_2^{n_2} \ldots}{n_1! n_2! \ldots} \prod_i a^{(x_i-2)n_i} \int \langle \phi_1(r_1^1)\phi_1(r_2^1)\ldots\phi_2(r_1^2)\ldots\rangle \prod d^2 r_i^j. \tag{6.21}$$

The microscopic length a appears in two places in the above expression: explicitly, as powers of a^{x_i-2}, and implicitly, because the integrals are in general UV divergent, and must be regularized with a cutoff, for example $|r_i^j - r_{i'}^{j'}| > a$. The integrals are also IR divergent. This may be corrected by, for example, putting theory in a finite box. The subsequent manipulations all involve the UV behavior, and are independent of such a cut-off.

To carry out the renormalization group, we change $a \to (1+dl)a$, and see how we must modify the $\{g_i\}$ in order to keep the form of the free energy intact. For an infinitesimal transformation ($dl \ll 1$), the contributions from the different ways a appears in (6.21) may simply be summed. The explicit dependence on a leads to a contribution $g_i \to (1+(2-x_i)dl)g_i$. Now consider the change in the expression

$$\int d^2r_1 d^2r_2 \ldots \langle \ldots \phi_i(r_1)\phi_j(r_2)\ldots\rangle H(|r_1-r_2|-a) \tag{6.22}$$

as $a \to (1+dl)a$. Using the operator product expansion, this may be written, after angular integration

$$-\int d^2r_1 \sum_k c_{ijk} \cdot 2\pi a^2 \, dl \, a^{-x_k} \langle \ldots \phi_k(r_1)\ldots\rangle. \tag{6.23}$$

We see that the term of order $g_i^{n_i} g_j^{n_j} g_k^{n_k} \ldots$ renormalizes the term of order $g_i^{n_i-1} g_j^{n_j-1} g_k^{n_k+1} \ldots$. On getting all the combinatorial factors straight, we find that the change in g_k, consistently to all orders in the expansion of the free energy, is $-\pi c_{ijk} g_i g_j$. Putting this all together, the lowest order renormalization group equations are

$$-\beta_k(g) \equiv \dot{g}_k = y_k g_k + \pi \sum_{i,j} c_{ijk} g_i g_j + O(g^3), \tag{6.24}$$

where $y_i = 2 - x_i$, and $\dot{g}_k = dg_k/dl \equiv -\beta_k(g)$. The higher order terms may also be calculated, with the knowledge of the four-point functions at the unstable fixed point, but a more sophisticated cutoff is required. As with the ε-expansion, on finds that the high enough order terms in the expansion in the g_k for fixed y_k have infrared divergences. This is cured in the same way, by considering a simultaneous expansion in both the g_k and the y_k.

The interesting feature of (6.24) is that, to this order, the renormalization group equations are gradient flows:

$$\dot{g}_k = \frac{\partial}{\partial g_k}\widetilde{C}(\{g\}), \tag{6.25}$$

where

$$\widetilde{C}(\{g\}) = \frac{1}{2}\sum_k y_k g_k^2 + \frac{1}{3}\pi\sum_{ijk} c_{ijk}g_i g_j g_k. \tag{6.26}$$

This makes it easy to calculate the function $C(\{g\})$ of Zamolodchikov's theorem [30]. Since it must have the same stationary points as $\widetilde{C}(\{g\})$, to this order they must be proportional:

$$C(\{g\}) = c + \alpha\widetilde{C}(\{g\}) + O(g^4). \tag{6.27}$$

The constant α may be fixed by calculating C perturbatively, using (6.7) and (6.11).

$$\begin{aligned} C &= c - \frac{3}{4}(2\pi)^2\sum_k y_k^2 g_k^2 \int_0^1 \frac{r^4}{r^{2(2-y_k)}}\frac{d(r^2)}{r^2} + \cdots \\ &= c - 3\pi^2\sum_k y_k g_k^2 + O(g^3), \end{aligned} \tag{6.28}$$

from which we see that $\alpha = -6\pi^2$.

These formulae allow us to calculate the value of c at the new fixed point, perturbatively in the y_k. Consider the simplest example with one relevant coupling g_1. For simplicity assume that $c_{11j} = 0$ for $j \neq 1$. Then there is a fixed point at $g_1 = g^* \equiv -(y_1/\pi c_{111}) + O(y^2)$. Substituting this into the above equations, we find that the value of c at the new fixed point is

$$\begin{aligned} &c - 6\pi^2\left(\frac{y_1 {g^*}^2}{2} + \frac{\pi c_{111} {g^*}^3}{3}\right) + \cdots \\ =&c - \frac{y_1^3}{c_{111}^2} + O(y_1^4). \end{aligned} \tag{6.29}$$

This agrees with another (more complicated) calculation [31] which uses the finite-size correction (3.7) to define c.

6.5 Application to models with $c < 1$

Let us consider the sequence of models with $c < 1$ corresponding to the diagonal modular invariant, which in Sec. (5.3) we argued were in the universality class of multicritical Ising models. The operator $\phi_{1,3}$ with $h = \bar{h} = h_{1,3}$ is always present in these models, and it has dimension

$$x_{1,3} = \frac{((m+1) - 3m)^2 - 1}{2m(m+1)} \sim 2 - \frac{4}{m} + O(m^{-2}) \tag{6.30}$$

as $m \to \infty$. Suppose we turn on this operator. The fusion rules imply that the only non-zero operator product expansion coefficients c_{11i} are those which couple $\phi_{1,3}$ to other operators in the column $r = 1$, which are all irrelevant, except for $i = 1$. Thus, we are in the situation of the calculation at the end of the last section.

The coefficient c_{111} may be found by taking the limit $m \to \infty$ in the results of Dotsenko and Fateev [32], and one finds

$$c_{111} = \frac{4}{\sqrt{3}}(1 + O(m^{-1})). \tag{6.31}$$

Thus the value of c at the new fixed point is

$$\begin{aligned} & c(m) - \frac{3}{4^2}\left(\frac{4}{m}\right)^3 + O(m^{-4}) \\ =& c(m) - 12/m^3 + O(m^{-4}) \\ =& c(m-1) + O(m^{-4}), \end{aligned} \tag{6.32}$$

to the order at which we are working. Thus, under the perturbation $\phi_{1,3}$, the renormalization group flows cross over to the theory with the next lowest value of c. This is consistent with the physical interpretation of these theories. In the Landau-Ginzburg picture (5.25), the operator $\phi_{1,3}$ corrsponds to $\Phi^{2(m-1)-2}$. When this is turned on, the term $\Phi^{2(m-1)}$ becomes irrelevant, and we would expect the theory to correspond to $c(m-1)$. At the new fixed point, the perturbation is irrelevant. From our renormalization group equations, we can see that its renormalization group eigenvalue is $-y \sim -4/m$. This is consistent with the operator moving to the position $(1,3)$ in the conformal grid, that of the leading irrelevant operator.

Another interesting example concerns the case $m = 5$. As discussed in Sec. (5), this corresponds to both the tetracritical Ising model and to the 3-state Potts model. The operator $\phi_{1,3}$, by the above argument, causes crossover in the tetracritical model to the tricritical Ising model with $c = \frac{7}{10}$. In the Potts model, this operator is the next-to-leading magnetic operator. Since the value of the change in c depends only on the correlation functions (which are the same for both models), the Potts model must also cross over to a theory with $c = \frac{7}{10}$. But modular invariance tells us that there is only one such theory –

the tricritical Ising model. This behaviour may be understood if we enlarge the parameter space of the tricritical Ising model, which usually has action

$$S = -K \sum_{\text{bonds}} s(r)s(r') + \mu \sum_{\text{sites}} s(r)^2, \tag{6.33}$$

where $s(r) = 1, 0, -1$, to include a term $K' \sum_{\text{sites}} s(r)^2 s(r')^2$. There is then a critical value of K' in terms of the other parameters for which this model has the Z_3 symmetry of the Potts model. The operator $\phi_{1,3}$ then breaks this Z_3 symmetry and gives a flow towards the usual tricritical Ising model with $K' = 0$.

6.6 Conserved currents away from criticality

In statistical mechanics or in quantum field theory, exact solvability is usually associated with the existence of an infinite number of conserved currents. In complex co-ordinates, a current $(J_{z\ldots}, J_{\bar{z}\ldots})$ is conserved if

$$\partial_{\bar{z}} J_{z\ldots} + \partial_z J_{\bar{z}\ldots} = 0, \tag{6.34}$$

where we include the dots to indicate that J may have rank greater than one. In conformal field theories, there are certainly an infinite number of such currents, for example $(z^{n+1}T(z), 0)$, whose line integrals give the Virasoro generators. Away from criticality, of course, these will not all be conserved. Only $T_{\mu\nu}$ itself, that is $(T, \frac{1}{4}\Theta)$ and $(\frac{1}{4}\Theta, \overline{T})$ are conserved because of translational invariance, and $\epsilon^{\sigma}_{\lambda} r^{\lambda} T_{\mu\sigma}$ is conserved as a consequence of rotational invariance.

However, there are many other conserved currents at criticality. An example is afforded by all the quasi-primary descendants of the identity operator $\mathbf{1}$, in the same conformal tower as the stress tensor $T \propto L_{-2}\mathbf{1}$. At level 3 we have $L_{-3}\mathbf{1} \sim [L_{-1}, L_{-2}]\mathbf{1} \sim \partial T$, so this is not quasi-primary. At level 4 there is the quasi-primary operator

$$T_4 \equiv (L_{-2}^2 - \frac{3}{5}L_{-4})\mathbf{1}, \tag{6.35}$$

which is in fact proportional to $:T^2:$, and so on. All of these operators depend only on z, and therefore give conserved currents $(T_4, 0)$, *etc.* There are other examples, *e.g.* the fermion $\psi(z)$ in the Ising model, and the spin 3 current $\phi_{4,1}$ with scaling dimensions $(3, 0)$ in the 3-state Potts models. All these latter conserved currents are signals of higher algebras in these models in addition to the Virasoro algebra.

Suppose that we now perturb the theory with a relevant operator as in (6.1). Do any of these currents continue to be conserved away from criticality? Generalizing our argument for the stress tensor to a generic current $(J,\ldots)$, we calculate perturbatively in λ:

$$\langle J(z,\bar{z})\ldots\rangle = \langle J(z)\ldots\rangle_{S^*} + \lambda \int \langle J(z)\phi(z_1,\bar{z}_1)\ldots\rangle_{S^*} d^2z_1 + \cdots . \tag{6.36}$$

This integral should be regulated with a cutoff $H(|z-z_1|-a)$. When this is done, $\partial_{\bar{z}} J$ is no longer zero. The term that survives in the limit $a \to 0$ is proportional to the coefficient of $(z-z_1)^{-1}$ in the operator product expansion of J with ϕ:

$$J(z)\phi(z_1,\bar{z}_1) = \cdots + \frac{1}{z-z_1} A^{(1)}(z,\bar{z}) + \cdots . \tag{6.37}$$

Note that the operator on the right hand side is referred to the point $(z,\bar{z})$, so this is a slightly modified version of the usual operator product expansion. If J has dimensions $(S,0)$ and ϕ has dimensions (h,h) as usual, then $A^{(1)}$ has dimensions $(h+S-1,h)$. In general, then, it will be a descendant of ϕ at level $S-1$. It is a standard exercise in BPZ technology to find the exact form of the operator. We then see that, to first order in λ,

$$\partial_{\bar{z}} J = \lambda A^{(1)} + \cdots . \tag{6.38}$$

The operator $A^{(n)}$ in the term of $O(\lambda^n)$ will have scaling dimensions $(S-n(1-h), 1-n(1-h))$. Since $h<1$, only a finite number of terms can exist. The question as to whether there is a conserved current now comes down to whether all the operators on the right hand side of (6.38) may be written as total derivatives with respect to z.

Let us consider some simple examples. First, the Ising model, with a thermal perturbation $\lambda \int \epsilon\, d^2z$. The conserved current at criticality is the fermion $\psi(z)$, with $S=\frac{1}{2}$. The operator $A^{(1)}$ has dimensions $(0,\frac{1}{2})$, and the only candidate for this is $\overline{\psi}$. Moreover, all the $A(n)$ with $n>1$ would have a negative dimension, and therefore must vanish. In this case, we see that the current is not in fact conserved away from criticality, but that the fermions do satisfy the massive Dirac equation $\partial_{\bar{z}}\psi \propto \lambda\overline{\psi}$, together with a similar equation for $\overline{\psi}$, which makes the model solvable.

Next [33], suppose that $J=T_4$, where T_4 was defined above. In that case $S=4$, and $A^{(n)}$ has dimensions $(4-n(1-h), 1-n(1-h))$. The only possibility for $A^{(1)}$ has the form

$$A^{(1)} = \left(\alpha L_{-1}^3 + \beta L_{-1}L_{-2} + \gamma L_{-3}\right)\phi, \tag{6.39}$$

where the constants α, β, γ are calculable. Now remember that $L_{-1} \sim \partial_z$. In general, because of the last term, (6.39) is not a total derivative with respect to z. However, suppose that $\phi = \phi_{1,3}$, that is, there is a null state at level 3. This means that all the correlation functions of the quasi-primary operator

$$\left(L_{-3} - \frac{2}{h+1} L_{-1} L_{-2} + \frac{1}{(h+1)(h+2)} L_{-1}^3 \right) \phi \tag{6.40}$$

vanish. Hence we may eliminate the last term in (6.39) in favor of the other two, and we see that the result is indeed a total derivative. Thus a conserved current does exist, to first order in λ.

Now look at candidate operators for $A^{(n)}$. The right scaling dimension of $A^{(n)}$ is < 1. Also, it must appear in the operator product expansion of $(\phi_{1,3})^n$. The only possibility is for it to have right scaling dimension zero, which means that $A^{(n)}$ has scaling dimensions $(3, 0)$, and that it will appear at order $n = (1-h)^{-1} = \frac{1}{2}(m+1)$. Such a contribution is therefore only possible if m is odd. In general, the only operator with such scaling dimensions is $L_{-1}^3 \mathbf{1} \sim \partial_z T$, so this term is also a total derivative. An exception is the 3-state Potts model ($m = 5$), where we saw that there is a primary operator with dimensions $(3, 0)$.

What does the conservation of $T_4 \sim : T^2 :$ mean? This is more easily understood in Minkowski space. When λ is turned on, the excitations corresponding to the operator $\phi_{1,3}$ acting on the vacuum become massive. Although the theory will continue to possess critical, massless, excitations, because of the fusion rules these will not appear as intermediate states in the correlation functions of the $\phi_{1,3}$ fields. Thus the long-distance behavior of this sector of the theory should be desecribed by a free, massive boson (*i.e.* (1.21) with a mass term proportional to $m^2\phi^2$). If we canonically quantize such a theory in terms of oscillators

$$\phi(x) = \int \frac{dk}{2\pi\sqrt{\omega_k}} \left[a_k e^{ikx} + a_k^\dagger e^{-ikx} \right], \tag{6.41}$$

where $\omega_k = \sqrt{m^2 + k^2}$, we find that, after normal ordering, $\int T dx = \sum_k (\omega_k - k) a_k^\dagger a_k$. Thus, the fact that T is conserved implies that in a scattering process, the sums of the left-momenta $k_L = \omega_k - k$ of the incoming particles is the same as that for the outgoing ones. Similarly for the right-momenta $k_R = \omega_k + k$. This, of course, just corresponds to energy-momentum conservation in the collision. If we now calculate $\int : T^2 : dx$, we find that it is proportional to $\sum_k k_L^3 a_k^\dagger a_k$. Thus, the sum of the *cubes* of the left- and right-momenta are also conserved. In an N-particle scattering process, this can be shown to imply that the S-matrix decomposes into a sum of terms in each of which a pair of particles scatters elastically while the other $(N-2)$ are spectators. This information is usually sufficient to calculate the S-matrix exactly.

This result has an obvious, although so far unexplored, connection with the exactly solved lattice models of ABF, discussed in Sec. (5.3). The operator $\phi_{1,3}$ in that case is precisely the one which takes the theory away from criticality tangent to the exact solution manifold.

Zamolodchikov [33] has argued further that in the above case there is an infinity of additional conserved currents corresponding to higher spin operators in the same conformal tower as the stress tensor. In addition, he has showed how theories with higher spin conserved primary currents (like the 3-state Potts model), may continue to possess these when perturbed away from criticality in suitable ways.

7. The spectrum of the corner transfer matrix in solvable models

7.1 Introduction

The corner transfer matrix (CTM) method of Baxter [15] is a particularly powerful method of obtaining exact information (in particular the one-point functions) in exactly solvable models both at and away from the critical point. Recently the Kyoto group [34] have identified and solved with this method a wide class of lattice models, one, in fact, for each conformal field theory which can be obtained using the GKO construction based on the $A_n^{(1)}$ algebras. A remarkable result is that the logarithm of the corner transfer matrix (to be defined below) is proportional to the Virasoro generator L_0, in that their spectra have the same spacing and degeneracies. This is quite unexpected, as the Virasoro algebra is supposed to be connected with the conformal symmetry, which operates only at the critical point, in the continuum limit. As we shall see, it has the consequence that the formulae for the one-point functions away from criticality are ratios of characters of a Virasoro algebra, with the modular parameter q representing, however, not the shape of a torus, but rather being related to the temperature.

At present, this observation lacks any general theoretical explanation. Moreover, even a descriptive account of the detailed results of the Kyoto group would take too long.

Therefore, in this lecture I intend to illustrate the correspondence in the very simplest case, that of the Ising model. Further details of the subtleties of the CTM method are discussed in Baxter's book [15].

7.2 Commuting transfer matrices

For convenience, we shall consider two independent Ising models on interpenetrating lattices rotated at 45° to the x and y axes. An essential part of the technique of the CTM is to consider anisotropic models, with couplings K_1 and K_2 as shown in Fig. 7.1. The doubled Ising model is a special case of the interaction-round-a-face (IRF) model in which spins (s_1, s_2, s_3, s_4) (in this case Ising spins) are placed around the four corners of an elementary square or plaquette, and the Boltzmann weight for the whole lattice is the product of the weights $w(s_1, s_2, s_3, s_4)$.

Figure 7.1 IRF model corresponding to the doubled Ising model.

In this case

$$w(s_1, s_2, s_3, s_4) = \mathrm{e}^{K_1 s_1 s_3}\, \mathrm{e}^{K_2 s_2 s_4}. \tag{7.1}$$

The doubled Ising model is in fact equivalent to the case $ab = cd$ of the 8-vertex model discussed briefly in Sec. (3.2).

Consider the row-to-row transfer matrix $\hat{T}(K_1, K_2)$ of this model. Baxter showed that the transfer matrices $\hat{T}(K_1, K_2)$ and $\hat{T}(K_1', K_2')$ *commute* if

$$\sinh 2K_1 \sinh 2K_2 = \sinh 2K_1' \sinh 2K_2' = k^{-1}, \quad \text{say.} \tag{7.2}$$

In terms of the Boltzmann weights $e^{\pm 2K_1}, e^{\pm 2K_2}$, this is an algebraic curve of genus one. It may be parametrized by elliptic functions: if we define $\operatorname{snh} u \equiv -i \operatorname{sn} iu$, where sn is the usual Jacobi elliptic function of modulus k, then

$$\begin{aligned} e^{-2K_1} &= \operatorname{snh} u / \operatorname{snh} \lambda, \\ e^{-2K_2} &= \operatorname{snh}(\lambda - u) / \operatorname{snh} \lambda. \end{aligned} \tag{7.3}$$

Provided that $k \operatorname{snh}^2 \lambda = 1$, one may show, using addition theorems for elliptic functions, that this parametrization satisfies (7.2). Moreover, and most important, the Boltzmann weights are meromorphic functions of u. In this parametrization, $k - 1$ measures the distance from criticality, and u measures the degree of anisotropy. The fact that transfer matrices with two different values of u, but the same k, commute, is a consequence of the star-triangle relation. This is illustrated in Fig. 7.2.

Figure 7.2 Star-triangle relation. The weights in the 3 squares have different values of u, but the same k. Only the central spin is summed over. The relation is valid if $u' = u + u''$.

Using this, we can consider three rows of the lattice corresponding to the product of transfer matrices $\hat{T}(u')\hat{T}(u)$, and move the square with $u'' = u' - u$ from right to left across the lattice. This will have the effect of interchanging u and u' at every position (Fig. 7.3).

All the models which have been solved using this method have the star-triangle property. The commutation of the transfer matrices means that their eigenvectors (but not of course their eigenvalues) are independent of u. This property may be generalised to the

Figure 7.3 Using the star-triangle relation to show that two transfer matrices commute.

case where the us depend on the x-coordinate. A similar argument shows that the transfer matrices $\hat{T}(u(x))$ and $\hat{T}(u(x)+u'')$ commute. Thus the eigenvectors depend only on the differences $u(x_1)-u(x_2)$.

7.3 Corner transfer matrix

The corner transfer matrix $\hat{A}$ is defined as the partition function of the upper right quadrant, when we fix the spins on the positive x-axis to particular values $\{s\}=(s_0,s_1,\ldots)$, and on the upper y-axis to the values $\{s'\}=(s'_0,s'_1,\ldots)$. The rows and columns of $\hat{A}$ are labelled by $\{s\}$ and $\{s'\}$ respectively. See Fig. 7.4. Note that $s_0=s'_0$, that is the spin at the origin is conserved under the action of $\hat{A}$.

In general $\hat{A}$ will depend on u. It will also depend on the boundary condition chosen at infinity. We choose to fix the spins at infinity in one of the ground states.

Figure 7.4 Definition of the corner transfer matrix.

We may also consider the CTM which rotates from the positve y-axis to the negative x-axis. This will be the same as $\hat{A}$ with the replacement $K_1 \leftrightarrow K_2$, which is equivalent to $u \to \lambda - u$. Now consider the partition function for the region $y \geq 0$, in which all the elementary squares in the upper right quadrant have an anisotropy parameter u, and all those in the upper left quadrant have anisotropy parameter $\lambda - v$. The spins on both halves of the x-axis are fixed to specified values $(\{s\}, \{s'\})$, as shown in Fig. 7.5.

Figure 7.5 Inhomogeneous lattice used to show addition formula for the CTM.

In terms of CTMs, this partition function is

$$\left(\hat{A}(u)\hat{A}(v)\right)_{s,s'}. \tag{7.4}$$

On the other hand, we could calculate this using the row-to-row transfer matrix $\hat{T}(u, \lambda - v)$. Since the lattice extends infinitely far in the y-direction, only the ground state $|0\rangle$ of $(-\ln \hat{T})$ will contribute. The result will be

$$\langle s, s'|0\rangle. \tag{7.5}$$

Howvever, by the star-triangle argument above, the state $|0\rangle$ will depend only on the *difference* $u - (\lambda - v)$. We see, therefore, that $\hat{A}(u)\hat{A}(v)$ depends only on the combination $u + v$. It follows that $\hat{A}(u)$ depends exponentially on u:

$$\hat{A}(u) = \mathrm{e}^{-u\hat{H}_c}. \tag{7.6}$$

In deriving this, we have been rather cavalier with normalizations and the thermodynamic limit. Nevertheless, as argued by Baxter, (7.6) is true up to a c-number factor.

The above result is not surprising in the continuum limit, close to isotropy ($u \simeq \frac{1}{2}\lambda$). In the isotropic case, we may consider a more general CTM which rotates through an angle θ. Rotational invariance implies that it should have the form

$$\hat{A}(\theta, \frac{1}{2}\lambda) = \mathrm{e}^{-\theta\hat{H}_c{}'}, \tag{7.7}$$

where the usual CTM is $\hat{A}(\pi/2, u)$. Perturbing u away from its isotropic value $\frac{1}{2}\lambda$ may be compensated, in the continuum limit, by rescaling $x + y$ and $x - y$ appropriately, This has the effect of modifying θ from $\pi/2$ to $\pi/2 + O(u - \frac{1}{2}\lambda)$. Thus, the exponential dependence on θ in (7.7) reflects the similar dependence on u in (7.6). However, it should be stressed that (7.6) is valid independently of whether we take the continuum limit.

7.4 The eigenvalues of $\hat{A}(u)$

We now use the fact that the elliptic function parametrization of the Boltzmann weights implies that the eigenvalues of $\hat{A}$ are periodic functions of u. There are of course two periods, real and imaginary. The real period has no physical meaning, merely adjusting the overall normalization of the CTM. However, the imaginary periodicity under $u \to$

$u + 4iK(k)$ (where $K(k)$ is the complete elliptic integral of the first kind), implies, if we look at (7.6) that

$$\textit{the eigenvalues of } \hat{H}_c \textit{ are } \pi/2K \textit{ times integers.} \tag{7.8}$$

Now comes the real *coup de main.* Since the spectrum of $(2K/\pi)\hat{H}_c$ is integrally spaced, it should be robust under any limit of the parameter k we choose to take. In particular consider the limit corresponding to the original Ising couplings $K_2 \to \infty$, with K_1 fixed. This corresponds to $k \to 0$ with u fixed. In that limit, $\mathrm{e}^{-2K_1} \sim \mathrm{e}^{-\pi u/2K}$. The matrix $\hat{A}$ is automatically diagonal, because the limit $K_2 \to \infty$ freezes all spins in the same left-slanting diagonal to be equal (see Fig. 7.6). The diagonals which intersect the x-axis at $x = j, j+2$ interact with a strength $(j+1)K_1$.

Figure 7.6 CTM in the limit $K_2 \to \infty$. Spins on same solid line are in the same state.

Thus

$$A(\{s\},\{s\}) = \exp\left(-(\pi u/4K)\sum_{j=0}^{\infty}(j+1)(1 - s_j s_{j+2})\right), \tag{7.9}$$

where we have normalized $\hat{A}$ so that its largest diagonal element is unity. (7.9) displays the advertised integer valued spectrum of $(2K/\pi)\hat{H}_c$. From (7.8), however, we conclude that (7.9) is valid in the basis in which $\hat{A}$ is diagonal for *all* $k < 1$.

We see that the problem has been reduced to that of solving a rather simple one-dimensional chain. This is a general feature of models solved using the CTM. In general these lead to *partition sums* which satisfy many beautiful identities (*e.g.* the Rogers-Ramanujan identities and generalizations thereof), and which in particular, are related to character formulae. Let us see how this works in the simple case under consideration.

7.5 Character formulae

The partition function for the whole lattice (keeping in mind the fact that we fix the spin at the origin s_0 and those at infinity to pre-assigned values) is given by

$$\begin{aligned} Z &= \mathrm{Tr}\,\hat{A}(u)\hat{A}(\lambda-u)\hat{A}(u)\hat{A}(\lambda-u), \\ &= \mathrm{Tr}\,\mathrm{e}^{-2\lambda\hat{H}_c}, \\ &= \sum_{\{s\}} q^{\frac{1}{4}\sum_{j=0}^{\infty}(j+1)(1-s_j s_{j+2})}, \end{aligned} \tag{7.10}$$

where $q = \mathrm{e}^{-2\pi\lambda/K}$. In this expression, it is clear that the even and odd values of j decouple, as they should. If we consider only the even sublattice, and let $j = 2k$, the sums may be simplified by defining $n_k = \frac{1}{2}(1 - s_{2k}s_{2k+2})$. This quantity is 1 or 0, depending on whether there is (is not) a domain wall in between k and $k+1$.

Consider first the case when s_0 has the same value as the spins at infinity. Then there must be an even number of domain walls. The partition function in this case is

$$Z_{++} = \sum_{\sum_k n_k \text{ even}} q^{\sum_{k=0}^{\infty}(k+\frac{1}{2})n_k} = \frac{1}{2}\prod_{k=0}^{\infty}(1+q^{k+\frac{1}{2}}) + \frac{1}{2}\prod_{k=0}^{\infty}(1-q^{k+\frac{1}{2}}). \tag{7.11}$$

In the same way Z_{-+} is given by the difference of the above two terms.

The above expressions are simple linear combinations of the characters χ_0 and $\chi_{1/2}$ of the Virasoro algebra with $c = \frac{1}{2}$ (without the factors of $q^{-c/24}$ included in the definition (4.14).) For example, from (4.24) we find

$$\chi_0(q) - \chi_{1/2}(q) = [F(q)]^{-1} \sum_{k=-\infty}^{\infty} \left(q^{\frac{(24k+1)^2-1}{48}} - q^{\frac{(24k+7)^2-1}{48}} + q^{\frac{(24k+13)^2-1}{48}} - q^{\frac{(24k+19)^2-1}{48}} \right), \tag{7.12}$$

where $F(q) = \prod_{n=1}^{\infty}(1-q^n)$. If we introduce $j = k/4$, the sum may be rewritten as

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{(6j+1)^2-1}{48}}. \tag{7.13}$$

Such sums may always be written as infinite products using the Jacobi triple product identity [18]

$$\sum_{j=-\infty}^{\infty} x^j Q^{j^2} = \prod_{n=1}^{\infty}(1-Q^{2n})(1+xQ^{2n-1})(1+x^{-1}Q^{2n-1}). \tag{7.14}$$

Taking $Q = q^{3/4}$ and $x = -q^{1/4}$ we then find that

$$\begin{aligned} \chi_0 - \chi_{1/2} &= \prod_n (1-q^n)^{-1} \prod_n (1-q^{3n/2})(1-q^{(3n-1)/2})(1-q^{(3n-2)/2}) \\ &= \prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}}). \end{aligned} \tag{7.15}$$

Similarly, one may show that

$$\chi_0 + \chi_{1/2} = \prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}). \tag{7.16}$$

We thus arrive at the very simple results

$$\begin{aligned} Z_{++} &= \chi_0(q), \\ Z_{-+} &= \chi_{1/2}(q). \end{aligned} \tag{7.17}$$

The appearance of the Virasoro characters implies that $\hat{H}_c$ is proportional to the generator L_0 of some Virasoro algebra. The origin and physical meaning of this algebra are at present unclear, although, in the case of the Ising model, an explicit representation of the other generators has been given by Otiyama and Thacker [35].

Using the above results, we may compute the spontaneous magnetization at the central site:

$$\langle s_0 \rangle = \frac{Z_{++} - Z_{-+}}{Z_{++} + Z_{-+}} = \prod_{n=1}^{\infty} \left(\frac{1-q^{n-\frac{1}{2}}}{1+q^{n-\frac{1}{2}}} \right). \tag{7.18}$$

The critical behavior occurs as $q \to 1-$. We may extract the singular behavior in this limit by using the properties of the characters under a modular tranformation, since, as $q \to 1-$, $\tilde{q} \to 0$. In this way, the character $\chi_{1/16}(\tilde{q})$ appears. This gives the familar Ising model result $\langle s \rangle \sim (T_c - T)^{1/8}$.

Although we have presented these results for the simplest of cases, it should be restated that these general features seem to appear in all models which are solvable by the CTM method [34]. The reason for this is left as an exercise for the ambitious student.

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