Other RG schemes

There are ~~almost~~ as many ways of doing the RG as papers written on the subject.
Historically, the Wilson scheme was very important. The basic idea is to start from the path integral

$$Z[J] = \int \mathcal{D}\phi \; e^{-\int [\frac{1}{2}(\nabla\phi)^2 + V(\phi) - J\phi] d^d r}$$

equipped with a cut-off Λ, and to vary $\Lambda \to \Lambda - \delta\Lambda$.
Idea is to split the Fourier transform $\tilde{\phi}(k)$ into components with $|k| < \Lambda - \delta\Lambda$ & those with $\Lambda - \delta\Lambda < |k| < \Lambda$ & 'integrate out' the latter. In practice, this can be done only perturbatively and is very tricky to higher orders.

Another perturbative scheme, which we shall develop in more detail because it applies to perturbations of more general (non-gaussian) fixed points, is sometimes called "Poor man's" renormalisation.

We consider an action of the form

$$S = S_0 + \sum_i \overset{\text{dim'less}}{\overset{\downarrow}{g_i}} \sum_r a^{x_i} \phi_i(r)$$

where S_0 already represents some known fixed point theory, and $\phi_i(r)$ are a set of scaling operators there.

Note we can also write $S \sim S_0 + \sum_i g_i \int \frac{d^d r}{a^{d-x_i}} \phi_i(r)$

The partition function is $\quad Z = \mathrm{Tr}\, e^{-S}$

and we can imagine expanding it in powers of the g_i:

$$Z = Z^* \Big[1 - \sum_i g_i \int \langle \phi_i(r) \rangle \frac{d^d r}{a^{d-x_i}} + \frac{1}{2} \sum_{ij} g_i g_j \int \langle \phi_i(r_1) \phi_j(r_2) \rangle \frac{d^d r_1 \, d^d r_2}{a^{d-x_i} \, a^{d-x_j}} \dots \Big]$$

where $\langle\ \rangle$ is wrt S_0^*.

- these integrals might be (indeed are) IR divergent: we put the system in a large box of size $L \gg a$.
- as written they may also diverge as $r_i \to r_j$: ~~cut~~ put the lattice "back in" with a cut-off $|r_i - r_j| > a$.

We now rescale $a \to a(1 + \delta l)$ (where $\delta l \ll 1$) & try to change the g_i so that Z stays the same.

a occurs in 3 places:

1) explicitly in a^{d-x_i}
2) in the cut-off $|r_i - r_j| > a$: looks like a hard-core gas
3) in the ratio L/a.

The first dependence is cancelled by rescaling

$$g_i \to (1 + \delta l)^{d-x_i} g_i \sim g_i + (d - x_i) g_i \delta l$$

The second we may write as

$$\int_{|r_1 - r_2| > a(1+\delta l)} = \int_{|r_1 - r_2| > a} - \underbrace{\int_{a(1+\delta l) > |r_1 - r_2| > a}}$$

In this piece $\phi_i(r_1)$ & $\phi_j(r_2)$ are close to each other!

we can use another important property of fixed pts (renormalised QFT)

the <u>Operator Product Expansion</u> (OPE)

$$\phi_i(r_1)\,\phi_j(r_2) = \sum_k{}' C_{ijk}(r_1 - r_2)\,\phi_k\left(\frac{r_1+r_2}{2}\right)$$

which is valid in the sense that the <u>correlators</u> with other operators far away are the same on the LHS & RHS.

If ϕ_i, ϕ_j, ϕ_k are rotational scalars then scale invariance implies

$$C_{ijk}(r_1 - r_2) = \frac{c_{ijk} \leftarrow \text{constant.}}{|r_1 - r_2|^{x_i + x_j - x_k}}$$

So we can write the extra term as

$$\frac{1}{2}\sum_{ij}\sum_k c_{ijk}\, a^{x_k - x_i - x_j} \underbrace{\int_{a(1+\delta l) > |r_1 - r_2| > a}}_{S_d\, a^d\, \delta l} \phi_k\left(\frac{r_1+r_2}{2}\right) \frac{d^d r_1\, d^d r_2}{a^{2d - x_i - x_j}}$$

where S_d = area of d-dim shell.

Finally if $\delta l \ll 1$ we can add the contributions

$$\frac{dg_k}{dl} = \underbrace{(d - x_k)}_{y_k \text{ as advertised}} g_k - \frac{1}{2} S_d \sum_{ij}{}' c_{ijk}\, g_i g_j + O(g^3)$$

These eqns. are <u>very general</u>

Application to ϕ^4 theory (Ising model).

In this case $$S = S_0 + g_2 \int \frac{d^d r}{a^{d-x_2}} \underbrace{\phi(r)^2}_{\phi_2} + g_4 \int \frac{d^d r}{a^{d-x_4}} \underbrace{\phi(r)^4}_{\phi_4}$$

where $S_0 = \int \frac{1}{2} (\nabla\phi)^2 \, d^d r$ is the gaussian theory.

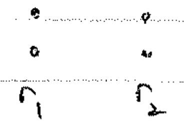
We can normalise ϕ so that $\langle \phi(r_1) \phi(r_2) \rangle_0 = \dfrac{1}{|r_1 - r_2|^{d-2}}$

and $\langle \phi^n(r_1) \phi^n(r_2) \rangle_0 \propto \dfrac{1}{|r_1 - r_2|^{n(d-2)}}$

so $x_2 = d-2$ $\quad x_4 = 2(d-2)$

The OPEs are very easy to work out using <u>Wick's theorem</u>

E.g. $\phi_2 \cdot \phi_2$



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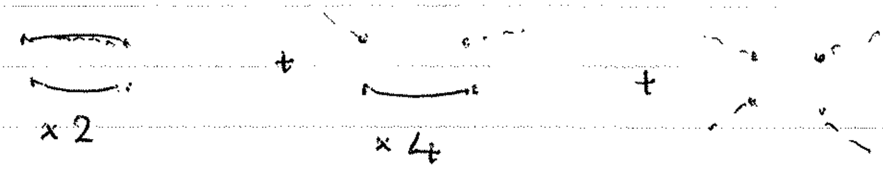
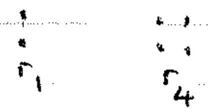


$$\phi_2(r_1)\,\phi_2(r_2) = \frac{2}{r_{12}^{2d-4}} + \frac{4}{r_{12}^{d-2}}\,\phi_2 + \phi_4 + \ldots$$

Similarly: $\phi_2 \cdot \phi_4$



$= 12\phi_2 + 8\phi_4 + \ldots$

$\phi_4 \cdot \phi_4$

$$= 24 + 96\phi_2 + \underset{\frac{(4\cdot 3)^2}{2}}{72\phi_4}$$

So the RG eqns are

$$\frac{dg_2}{dl} = 2g_2 - 4g_2^2 - 2\cdot 12 g_2 g_4 - 96 g_4^2 + \ldots$$

$$\frac{dg_4}{dl} = (4-d)g_4 - g_2^2 - 2\cdot 8 g_2 g_4 - 72 g_4^2 + \ldots$$

If we now put $d = 4 - \epsilon$ we see that there is a fixed point at $g_2 = O(\epsilon^2)$, $g_4 = \frac{\epsilon}{72} + O(\epsilon^2)$

At this fixed pt. $g_4' = g_4 - g_4^*$ is irrelevent but

$$\frac{dg_2}{dl} = \left(2 - \frac{24}{72}\epsilon + O(\epsilon^2)\right) g_2$$

so $y_2 = 2 - \frac{\epsilon}{3} + O(\epsilon^2)$

leading to $\nu = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2)$.