Scaling and Renormalization in Statistical Physics
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Hints for the Exercises

The exercises in this book were designed with a number of aims. Some merely repeat the arguments of the text, with variations, to ensure you have understood them. Some are open-ended, intended to guide you into the literature should you decide to follow them further. Some are deceptively simple, others technically difficult. Over the years, I have received a number of queries, particularly about some of the latter. The hints below should help.

John Cardy, Oxford, 2002.

Chapter 1.

1.1 Note that the variables $\frac{1}{2}(1+s(r))$ take the values 0 or 1 according to whether there is (is not) a particle of type A. So the interaction between the A atoms can be written $\sum_{rr'} J_{AA}(r-r')\frac{1}{2}(1+s(r))\frac{1}{2}(1+s(r'))$. Similarly for the others. Expanding this out, you'll find the hamiltonian for an Ising model, with some effective exchange interaction $J(r-r')$ and some effective magnetic field. Ising critical behaviour should occur when the effective magnetic field vanishes, at some critical temperature, as long as $J>0$.

1.2 The first part is standard. On a triangular lattice you can't perform the same trick. If you try to find the lowest energy state you'll find there are many of them. This is an example of a *fully frustrated system*. In fact, at zero temperature, this model is critical, in the sense that the spin-spin correlations decay as a power of the separation.

1.3 This question is supposed to test how you would answer an experimentalist who came and said "But my system isn't a spin-$\frac{1}{2}$ Ising model, it has spin (eg) $\frac{5}{2}$." You can't map the problems exactly, but you can, for example, compare the first terms in the high-temperature expansion of their susceptibility in powers of $1/k_BT$.

1.4 Have some coins (eg pennies, yen, or whatever is your local currency) available and draw some pictures of the square lattice so that the diameter of a coin is slightly larger than the lattice spacing (but smaller than $\sqrt{2}\times$ the lattice spacing.) You'll see that there are two distinct configurations which minimise the energy. Even though you cannot map this problem exactly onto a nearest neighbour Ising model, the problems have the same symmetry. The last part of the question is to test your knowledge of Bragg scattering. What is the size of the effective unit cell above and below the critical temperature?

Chapter 2.

2.1 Feynman's inequality comes from $\langle e^X\rangle \geq e^{\langle X\rangle}$, taking $\langle\cdot\rangle = \langle\cdot\rangle_{\mathcal{H}'}$ and $X = -\mathcal{H}+\mathcal{H}'$. See Feynman, *Statistical Mechanics*, p. 233 (Benjamin).

2.3 You can express the relevant integrals as Bessel functions of the form $\int \cos^n\theta e^{h\cos\theta} d\theta$, but, since you only need the behaviour for small h to understand the transition, this merely confuses things.

2.4 If the uniform magnetisation is M and the staggered magnetisation is M', then, in the mean-field approximation, the magnetisation on one sublattice is $\frac{1}{2}(M+M')$ and on the other is $\frac{1}{2}(M-M')$.

2.5 As in the text, to get the Landau exponents you do not need the explicit forms for all the coefficients in the expansion of the free energy, just which ones vanish.

2.6 If you think about it, all that changes relative to the case considered in the text is that (2.10) becomes $f \sim tM + O(M^6)$, so that the mean-field t-dependence of the free energy is modified.

2.7 (2.15) still holds, but now $M \to M_0$ as $r \to \infty$. This looks too hard to solve, but once again you can linearise, this time about $M = M_0$, for large r.

2.8 This question is in the pure spirit of Landau – what can you get, making the weakest possible assumptions? Think of $\mathbf{Q}$ as a symmetric traceless 3×3 matrix, and write down all the rotationally invariant quantities you can think of. For example, at second order there is $\mathrm{Tr}\,(\mathbf{Q})^2$. Are there any at third order? It doesn't matter that you don't know the microscopic values of the coefficients.

Chapter 3.

3.1 This is sometimes called a *decimation* transformation. You can make the algebra hideous or easy depending on how smart you are. (For example if you write $e^{Ks_1s_2} = \cosh K(1 + \tanh K s_1 s_2)$ and use the fact that $\mathrm{Tr}\, s_i^n = 0$ or 2 depending on whether n is odd or even, then the traces are much easier.) Make sure you have all the fixed points, including the antiferromagnetic one.

3.2 Ditto: write $e^{K\delta_{s_1s_2}}$ as $1 + (e^K - 1)\delta_{s_1s_2}$.

3.3 If the two largest eigenvalues of the transfer matrix are $\Lambda_0 > \Lambda_1$, then $f = -ln\Lambda_0$ and $\xi^{-1} = \ln(\Lambda_0/\Lambda_1)$.

3.4 This is one of those open-ended questions. To check your answers, you might want to look at the article by V. Privman, P. C. Hohenberg, and A. Aharony, in *Phase Transitions and Critical Phenomena* vol. 14, eds. C. Domb and J. Lebowitz (Academic Press, London 1991).

3.5 Just work out $dg_i'/d\ell$ as an expansion in the gs, and express the gs back as an expansion in the g's, working throughout to second order. See F. Wegner in *Phase Transitions and Critical Phenomena* vol. 6, eds. C. Domb and M. S. Green (Academic Press, London 1976).

3.6 Go through the manipulations on pp. 56-57, and integrate by parts until you get an integral of the form $\int_{u_t}^{\infty} s^{-1} g^{(n)}(s) ds$. Then estimate this (carefully!)

Chapter 4.

4.1 This is a matter of writing the scaling form for the free energy as in (3.28), and deciding what variable you are allowed to expand in. The point is that both scaling variables are proportional to t as you move away from the critical point, at fixed vacancy potential. The second part comes from differentiating the scaling form of the free energy with respect to the correct variable.

4.2 The easiest way to compute the susceptibility is as $\sum_{i,j}\langle s_i s_j\rangle$, where the correlation function can be computed exactly from the high-temperature expansion, that is, writing $e^{Ks_js_{j+1}} \propto (1+xs_js_{j+1})$, expanding out, and taking the trace, when all but a few terms vanish. The answers will depend on the boundary conditions.

4.3 The quantities $\xi(L,t)$, $\xi(t)$ and L all have the dimensions of length, where $\xi(t) \sim |t|^{-\nu}$ is the correlation length at reduced temperature t in the infinite system. So it should be easy to write down a scaling relation between them, just on the basis of dimensional analysis. Now suppose you have found numerical data on $\xi(L,t)$ for various values of L and t. What is the most efficient and accurate way to extract ν? When you have done your best, look at M. N. Barber, in Vol. 8 of Domb and Green.

4.4 This involves generalising equations like (4.15,4.16) to the logarithmic case.

4.5 The hint is already given in the question: this is still a delicate calculation! See I.B. Ferreira, A. R. King, V. Jaccarino and J. Cardy, Phys. Rev. B **28**, 5192, 1983.

4.6 For the second part, see E. Brézin and J. Zinn-Justin, Nucl. Phys. B **257**, 867, 1985.

5.1 You can save yourself a huge amount of labour here by realising, in analogy with the remark at the bottom of p. 98, that only a few OPE coefficients need actually be computed. The relevant operators now are ϕ_j with $1 \le j \le 4$. The last part goes in analogy with Sec. 5.6.

5.2 As on p. 105, you can simply generalise the OPE coefficients by first computing the $O(n)$ term, then comparing with the case $n = 1$. The large n calculation is analogous to that on p. 106, except that now $\xi^{-2} = t + O(\langle \mathbf{S}^2\rangle^2)$.

5.4 This requires that (a) you first do a calculation like that in Sec. 5.6: this is actually easier, because $\xi \sim e^{\ell_0}$; then (b) you look at (5.62) when $d = 4$.

5.5 This one is a bit more sophisticated: note that the operator as written transforms irreducibly under O(3) (according to the $l = 2$ representation), so that its OPE with $(\mathbf{S}^2)^2$ can only have the same combination of quantities on the RHS. This makes the calculation a lot easier: you only have to work out the coefficient of (say) S_z^2 on the RHS and the others are all determined. Once you have the OPE coefficient, the rest is as before.

5.6 This is a nice example. The only danger is in trying to expand in powers of ϵ at the wrong point in the calculation: don't do this: just assume that the RG equations are exact.

Chapter 6

6.1 The first part just generalises the calculation on p. 115 to the case $p \neq 1$. Then use (3.55). To work out the nature of the phase for $T < T_p$, observe that the coefficient of the $\cos p\theta$ term grows large, so that only certain values of θ are important. The last part involves comparing T_p with T_{KT}. The classic reference for all of this is José, Kadanoff, Kirkpatrick and Nelson, Phys. Rev. B **16**, 12, 1977.

6.2 The equations of Sec. 6.3 now become only true mod p. This used to be an industry: see R. Savit, Rev. Mod. Phys. ???

6.3 Just realise that K now has non-trivial dimensions so that, when it is expressed in terms of J, there is now a non-zero power of a. This gives a non-zero term in $dK/d\ell$. Now just assume (and justify afterwards, for small ϵ) that the other terms in the Kosterlitz equations are unmodified. See J. Cardy and H. Hamber, Phys. Rev. Lett. **45**, 499, 1980.

6.4 The hint in the question should be enough. This is the simplest way I know of to get the first order correction to the magnetic exponent, but see, eg, Amit's book (second edition) for the field-theoretic method. The analogous result for the cubic symmetry breaking term in part two of the question has never been published to my knowledge.

6.5 One of those disarmingly simple questions if you know how – just write down the form of the first couple of terms in the beta-function!

Chapter 7

7.1 Start with (7.3) but now impose the boundary condition that $M \to M_0$ as $z \to \infty$.

7.2 Starting from the RG equations (7.8), linearised about the fixed point, write down the analogue of (3.25) but for the *total* free energy, then break it up into parts proportional to V, A, etc, thus deriving scaling forms for the singular parts of f_b, $f_s, \ldots$. Then differentiate wrt the *bulk* parameters.

7.3 In the bulk, the calculation of the 2-point correlation function is given on p. 115. You can think of this as the exponential of the energy between a + charge at r_1 and a - charge at r_2. With the boundary, each of these charges has its image (either of the same or opposite sign, depending on the boundary conditions.) So altogether there are 4 charges and you have to work out their interaction energy (which is just the sum of the pairwise energies) and exponentiate the result. Now you can let r_1 and r_2 approach the boundary (you should keep them a distance $O(a)$ away, otherwise the answer might diverge), then you can work out the dependence on r_{12}.

7.4 The answer is simple, but then try to understand what happens as J_s gets large. Also, what happens for $d = 3$ in the XY model?

7.5 This involves solving Laplace's equation with Dirichlet boundary conditions and a unit charge near (not at) the apex of the wedge. You need only work out the behaviour of the solution as $r \to \infty$.

7.6 This is a hard problem: you have to modify the arguments of pp. 142-3 at several points. To check your answer, look in the review by Diehl in Domb and Lebowitz, vol. 10.

Chapter 8

8.1 For a given x, you have to work out the probability that a given site j is occupied and has exactly n_1 spins to the left of it until the first vacancy is reached, and n_2 spins to its right. This should be multiplied by the susceptibility of this site, $\sum_i \langle s_i s_j \rangle$, where the correlation function is the one for a *finite* chain (see Ex. 4.2). Now sum over n_1 and n_2.

8.2 The operator $E_a(r)E_b(r)$ on p. 149 gets replaced by $\sum_{r'}(r-r')^{-d-\sigma}$ $E_a(r)E_b(r')$. Work out its 2-point function, as in Sec. 4.3, to get at its scaling dimension and hence its RG eigenvalue. The second part is similar, but the impurities are now perfectly correlated in the imaginary time direction.

8.3 This is a delicate calculation. You have to use a version of (3.62) with another variable included, representing Δ. You have to integrate (8.17) and (8.19) first. It is worth doing this for general n – you should find that the $\ln\ln|t|$ behaviour is specific to $n = 0$.

8.4 For $n = 0$ there is degeneracy in the one-loop RG equations, and it is necessary to go second order. There is then a fixed point, of $O(\epsilon^{1/2})$, first found by Khmelnitski. However, this is not the question. You are asked to consider n replicas of the $O(m)$ model, so now there are two labels to think about. Have fun!

8.5 After replicating, and averaging over $\phi(r)$, you should end up with an effective interaction $\propto h_p^2 \sum_{a\neq b} \cos p(\theta_a - \theta_b)$. Now work out the two-point function of this operator, in the limit when $h_p = 0$.

8.6 See J. Cardy, J. Phys. A, **25**, L201, 1992.

8.7 Open-ended. See the Aharony, Hohenberg and Privman review referred to above.

Chapter 9

9.1 If the jth segment is represented by a vector $\mathbf{a}_j$ of length a, then, if one end is fixed at the origin, the position of the end of the jth monomer is $\mathbf{r}_j = \sum_{i=1}^{j} \mathbf{a}_i$. The mean square end-to-end distance is then r_N, and the radius of gyration squared is $(1/N^2)\sum_{i<j}(\mathbf{r}_i-\mathbf{r}_j)^2$. Their averages are easy to work out since $\langle\mathbf{a}_i\cdot\mathbf{a}_j\rangle = a^2\delta_{ij}$. The calculation for a closed loop is slightly more tricky since $\mathbf{r}_N=\sum_{i=1}^N\mathbf{a}_i=0$. Impose this constraint using the exponential integral representation of a delta-function.

9.2 (9.2) gets replaced by $u'a^{2d}\sum_{i,j,k}\delta^{(d)}(\mathbf{r}_i-\mathbf{r}_j)\delta^{(d)}(\mathbf{r}_i-\mathbf{r}_k)$.

9.3 Label two points on a given loop. Then you get a pair of mutually avoiding walks between the two points. For each distinct loop, how many distinct such pairs of walks do you get? If the lengths of the walks are N_1 and N_1, and the number of such pairs is $p(N_1,N_2)$, in the first part you worked out $\sum_{N_1+N_2=N}p(N_1,N_2)$. Now write a scaling form for $p(N_1,N_2)$ when N_1 and N_2 are both large and $x=N_1/N_2$ is fixed. Convince yourself that the scaling function is peaked around $x=1$ with a width of $O(1)$. Use this to compare with $\sum_{N_1+N_2=N}p(N_1,N_2)$ and hence work out $p(N/2,N/2)$.

9.4 The energy operator in the O(n) model is $E(r)\sim\mathbf{s}(r)\cdot\mathbf{s}(r')$, defined on the bond (r,r'). Show that E gives the local monomer density in this model. Write down the scaling form for $\langle sEEs\rangle$ using (3.59). q couples to the distance r_{12} between the two E operators, so that large q corresponds to small r_{12}. In this region you can use the form of the OPE $E(r_1).E(r_2)$. The leading non-trivial term on the RHS is again E, and using (5.6) you can then work out the dependence on r_{12} and hence on q.

9.5 The new feature is that you have to take the inverse Laplace transform wrt the fugacity x in order to get predictions for large but fixed N.

9.6 You have to work out the OPE of $\phi^{(p)}$ with the $\phi_4=(\mathbf{S}^2)^2$ operator of the O(n) model. Two of the Ss in this contract onto two of the Ss in $\phi^{(p)}$. How many ways are there of doing this?

9.7 The first part just involves taking the inverse Laplace transform wrt t. Now the specific heat is given by an integral over the two-point function of the energy density. Assume that dimensional reduction holds for this, ie as a function of r_{12} it is the same as the two-point function in a model in $d-2$ dimensions in which conventional hyperscaling holds, so that you can write down a scaling form for this 2-point function whose integrated form in $d-2$ dimensions agrees with hyperscaling. Now integrate up in d dimensions to get the specific heat of the branched polymer model.

Chapter 10

10.2 Such higher-order FD relations are not often discussed in the literature, but there is an infinite number of them!

10.3 Obviously such terms have to involve higher powers of S and/or derivatives. An example is $S^2\nabla^2 S$.

10.4 You have to argue that the limits $u \to 0$ and $\omega \to 0$ do not commute.

10.5 Write down dynamic scaling for G and compare this with the expected behaviour for $p > p_c$.

10.6 Just a matter of modifying the argument on p. 201. You should find $d_c = 2$.

Chapter 11

11.1 The mathematical way to do this is to show that the metric $ds^2 = \sum_\mu (dr^\mu)^2$ can be written as some scalar function of r' times $\sum_\mu (dr'^\mu)^2$.

11.2 The inversion $\mathbf{r} \to \mathbf{r}'$ with $\mathbf{r}' = (\mathbf{r} - \mathbf{R}_0)/|\mathbf{r} - \mathbf{R}_0|^2$ sends the sphere $|\mathbf{r}| = |\mathbf{R}_0|$ to a plane a distance $1/(2R_0)$ from the origin. Then apply (11.3). See T. W. Burkhardt and E. Eisenriegler, J. Phys. A **18**, L83, 1985.

11.3 Use $z \to z' = (L/\pi)\ln z$ in (11.11), then the ideas of Chapter 7.

11.4 Use $z \to z' = z^{\theta/\pi}$ in (11.11), then the ideas of Chapter 7.

11.6 Use the standard statistical mechanics formulas for the grand canonical ensemble, with a dispersion relation $\epsilon_k = |k|$.