Version 23 April 2001

Lectures on Conformal Invariance and Percolation*

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Abstract

These lectures give an introduction to the methods of conformal field theory as applied to deriving certain results in two-dimensional critical percolation: namely the probability that there exists at least one cluster connecting two disjoint segments of the boundary of a simply connected region; and the mean number of such clusters. No previous familiarity with conformal field theory is assumed, but in the course of the argument many of its important concepts are introduced in as simple a manner as possible. A brief account is also given of some recent alternative approaches to deriving these kinds of result.

*Lectures delivered in "New Trends of Mathematical Physics and Probability Theory", Chuo University, Bunkyo-ku, Tokyo, March 5-6, 2001.

1 Introduction.

The percolation problem has for many years been of great interest to theoretical physicists and mathematicians, in part because it is so simply stated yet so full of fascinating results. It embodies many of the important concepts of critical phenomena, yet is purely geometrical in nature.

Percolation studies the clustering properties of identical objects which are randomly and uniformly distributed through space. In lattice *bond* percolation, the links of a regular lattice, of edge length a, are either *open* or *closed.* In the simplest version of the model, the open bonds are independently distributed with a probability p for each to be open (and $1-p$ to be closed.) In *site* percolation, the bonds are all assumed to be open, but now each site is open with probability p. In both cases we study the statistical properties of *clusters* of neighbouring open bonds and sites. When p is small, the mean cluster size is also small, but, in more than one dimension, there is a critical value p_c of p, called the percolation threshold, at which the mean cluster size diverges. For $p>p_c$ there is a finite probability that a given site belongs to an infinitely large cluster.[1]

In these lectures, we shall be concerned with properties of the *continuum limit* of percolation. This may be defined as follows (we consider two dimensions from now on): consider a finite region $\mathcal{R}$ of the plane, bounded by a curve Γ. Consider a percolation problem on a sequence of lattices $\mathcal{L}_a$ covering $\mathcal{R}$, constructed in such a way that the lattice spacing $a\to 0$ keeping the size of $\mathcal{R}$ fixed. Obviously when we do this, such quantities as the total number of clusters in $\mathcal{R}$ will diverge as $a\to 0$, so we need to identify some suitable quantities which might have a finite limit. An example is afforded by the *crossing probabilities*. Suppose for simplicity that $\mathcal{R}$ is simply connected,

[1]There are also so-called continuum versions of percolation, for example the clusters formed by hard spheres of a given radius a which are distributed independently so that they may overlap. In all cases, however, a finite microscopic length a is necessary to define the notion of clustering.

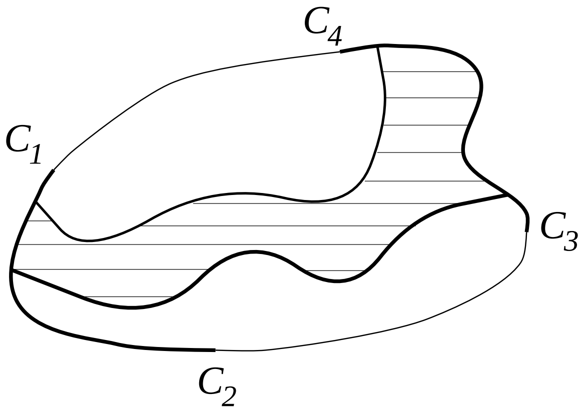

Figure 1: A crossing cluster from γ_1 (C_1C_2) to γ_2 (C_3C_4).

and let γ_1 and γ_2 be two disjoint segments of Γ (see Fig. 1). Then a *crossing event* is a configuration of bonds (or sites) on the lattice $\mathcal{L}_a$ such that there exists at least one cluster, wholly contained within $\mathcal{R}$, containing both at least one point of γ_1 and of γ_2. Let $P(\gamma_1, \gamma_2; \mathcal{L}_a)$ be the probability of this event. Then the following statements are all conjectured to be true:

- $P(\gamma_1, \gamma_2; \mathcal{L}_a)$ has a finite limit, denoted by $P(\gamma_1, \gamma_2)$, as $a \to 0$, which (under a broad class of conditions) is independent of the particular form of $\mathcal{L}$, of the precise way in which $\mathcal{L}$ intersects the boundary Γ, and of whether the microscopic model is formulated as bond, site, or any other type of percolation, as long as there are only short-range correlations in the probability measure.[2]

- $P(\gamma_1, \gamma_2)$ is invariant under transformations of $\mathcal{R}$ which are conformal in its interior (but not necessarily on its boundary Γ).

- The Riemann mapping theorem allows us to conformally map the interior of $\mathcal{R}$ onto the interior of the unit disc $|z| < 1$ of the complex

[2]Of course, this limit is interesting only at $p = p_c$. For $p < p_c$, $\lim_{a\to 0} P = 0$, because all clusters are finite (in units of a), while for $p > p_c$ the infinite cluster always spans, so the limit is 1.

plane. Suppose that the ends of the segments are thereby mapped into the points (z_1, z_2, z_3, z_4) (see Fig. 1 for the labelling). Then $P(\gamma_1, \gamma_2)$ is a function only of their cross-ratio

$$\eta \equiv \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \quad , \tag{1}$$

and has the explicit form

$$P = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}\, \eta^{\frac{1}{3}}\, {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; \eta) \tag{2}$$

where ${}_2F_1$ is the hypergeometric function.

- Moreover, if the random variable $N_c(\gamma_1, \gamma_2; \mathcal{L}_a)$ denotes the total number of distinct clusters which cross from γ_1 to γ_2, then the whole probability distribution of N_c also has a finite limit as $a \to 0$ and is conformally invariant, depending only on η. In particular the mean number of such crossing clusters is

$$E[N_c] = \tfrac{1}{2} - \tfrac{\sqrt{3}}{4\pi}\left[\ln(1-\eta) + 2\sum_{m=1}^{\infty} \frac{\Gamma(\frac{1}{3}+m)\Gamma(\frac{2}{3})}{\Gamma(\frac{2}{3}+m)\Gamma(\frac{1}{3})} \frac{(1-\eta)^m}{m}\right] \tag{3}$$

Equations (2,3) are just two among many similar results which may been obtained using methods of conformal field theory developed by theoretical physicists[1, 2, 3]. At first sight, the appearance of formulas like these may seem quite mysterious to those used to thinking about percolation as a lattice problem. It is certainly true that the methods originally used to derive them are not, so far, mathematically rigorous.[3] But there is no doubt that they are correct - (2) has been numerically tested to great precision in a number of cases[4], and, moreover, once the existence of a conformally invariant continuum limit is accepted, the formulas follow from very classical mathematical methods. It is the purpose of these lectures to give some idea to a

[3] S. Smirnov[6] has recently given a proof of (2) using other methods.

non-specialist audience of how these kinds of results arise. Inevitably I shall not be able to cover all the details, but hopefully the lectures will provide a basis from which to explore the literature further[5].[4] For reasons of time I will focus on the derivation of the above formulae, which is mostly my own work, as well as mention some other recent alternative derivations. This is not to overlook the work of others on other important results in percolation, particularly that of Nienhuis, Duplantier, Saleur and others on the Coulomb gas approach, which is better suited to systems without boundaries.

2 Percolation, the random cluster model, and the Potts model.

The emphasis in this course will be on the analogy between percolation and conventional critical behaviour in spin systems. This is through a well-known mapping first discovered by Fortuin and Kastelyn[7]. The Potts model is a generalisation of the Ising model in which the spins $s(r)$ at each site of a lattice take the values $(1, 2, \ldots, Q)$, where, initially, Q is an integer larger than 1. The energy is the sum over all nearest neighbour pairs (r', r'') of sites (i.e. a sum over all bonds) of $-J\delta_{s(r'),s(r'')}$. Thus the partition function is

$$Z = \mathrm{Tr}\ \exp(\beta J \sum_{r',r''} \delta_{s(r'),s(r'')}) \tag{4}$$

Apart from an overall unimportant constant this may be rewritten as

$$Z = \mathrm{Tr} \prod_{r',r''} \Big((1-p) + p\delta_{s(r'),s(r'')}\Big) \tag{5}$$

where $p = 1 - e^{-\beta J}$.

[4]The main text includes only those concepts and arguments needed to arrive at the final conclusion. Footnotes will indicate how these ideas fit within the more general framework of conformal field theory.

Now imagine expanding out the product. If there are B bonds on the lattice there will be 2^B terms in this expansion. Each term may be associated with a configuration in which each bond of the lattice is open (if we choose the term $\propto p$), or closed (if we choose the term $\propto (1-p)$). Sites connected by open bonds form clusters, and the Kronecker deltas force the all the spins in a given cluster to be in the same state. When we trace over the spins, each cluster will have only one free spin, so will give a factor Q. Thus we can write Z as a sum over configurations $\mathcal{C}$ of open bonds:

$$Z = \sum_{\mathcal{C}} p^{|\mathcal{C}|}(1-p)^{B-|\mathcal{C}|}\, Q^{N(\mathcal{C})} \tag{6}$$

where $N(\mathcal{C})$ is the number of distinct clusters in $\mathcal{C}$. Note that in this form, at least for a finite lattice, Z is a polynomial in Q and therefore its definition may be extended to non-integer values of Q.

The weights in (6) define the *random cluster model.* Of course, in percolation, each cluster is weighted with a factor 1, so it corresponds to $Q = 1$. In that case, the sum is simply over all possible configurations weighted by their probabilities, so $Z(Q=1) = 1$. However, there is nontrivial information in the correlation functions. For example, the probability that sites r_1 and r_2 are in the same cluster is given by the limit as $Q \to 1$ of

$$\langle \delta_{s(r_1),a}\delta_{s(r_2),a}\rangle - \langle \delta_{s(r_1),a}\delta_{s(r_2),b}\rangle \tag{7}$$

where a and b are any two different Potts states.[5] From (6) we can also calculate quantities like the mean total number of clusters by differentiating with respect to Q:

$$E[N] = (\partial/\partial Q)|_{Q=1}\, Z(Q) \tag{8}$$

It is important establish how quantities like the crossing probabilities may be related to those in the Potts model. For this we need to bring in the

[5]This follows from a similar argument to that which gives the crossing probabilities in terms of partition functions, described later.

notion of boundary conditions. Consider a Potts model defined on a lattice $\mathcal{L}$ which covers the region $\mathcal{R}$ of the plane, as described in the Introduction. We consider the boundary of $\mathcal{L}$ as being the set of sites which lie just outside $\mathcal{R}$ but are adjacent to sites within $\mathcal{R}$. These boundary sites form a discrete approximation to the boundary Γ. On these boundary sites, there are two simple and natural boundary conditions we might impose on the Potts spins: *free*, which means that we sum freely over them in the partition function; and *fixed*, in which case they are all fixed into some particular Potts state, say a.

Notice that if we take the same boundary condition on the whole boundary, then in the limit $Q \to 1$ partition functions with either free or fixed boundary conditions become the same, $Z = 1$, since all spins must be in the same state! But we can get something nontrivial if we allow the boundary conditions to be different on different parts of the boundary. In particular, consider the geometry of Fig. 1, and suppose the boundary spins are fixed on the segments γ_1 and γ_2, and free on the rest of the boundary. By the permutation symmetry of the Potts states, there are two possible different cases: when the fixed states on γ_1 and γ_2 are the same, for example a; or when they are different, say a and b. Let us denote the partition functions in the two cases by $Z_{aa}(Q)$ and $Z_{ab}(Q)$ respectively. Now each configuration $\mathcal{C}$ either has, or does not have, a cluster which spans between γ_1 and γ_2 (see Fig. 2). Those which do not have a spanning cluster contribute to both partition functions (with the appropriate weights), but configurations which do cannot contribute to $Z_{ab}(Q)$, since the existence of the cluster would force spins on the two segments to be in the same state, which is not true by assumption. Thus we have a simple relation for the crossing probability, on

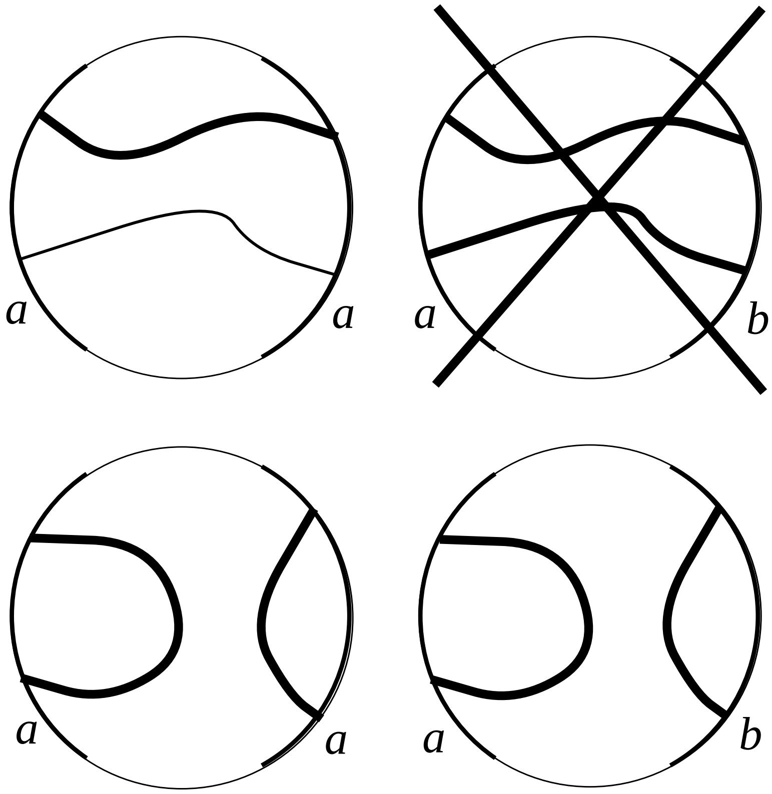

Figure 2: Crossing events contribute to Z_{aa} but not to Z_{ab}, while non-crossing events contribute equally to both.

the lattice:[6]

$$P(\gamma_1,\gamma_2;\mathcal{L}) = \lim_{Q\to 1}\left(Z_{aa}(Q) - Z_{ab}(Q)\right) \tag{9}$$

Remember that each of these is a polynomial in Q, so that Z_{ab} makes perfect sense at $Q=1$, even though it might seem that we need at least two states to define it.[7]

With only slightly more effort we can relate the mean number of distinct crossing clusters $E[N_c]$ to partition functions. Consider once again the geometry of Fig. 1, but now subdivide the clusters into those which touch neither γ_1 nor γ_2, those which touch γ_1 but not γ_2, those which touch γ_2 but not γ_1, and finally the crossing clusters, which touch both γ_1 and γ_2. Denote the number of each such cluster in each configuration by N_0, N_L, N_R and N_c respectively (see Fig. 3.) Since clusters which touch a boundary where the Potts spins are fixed are counted with weight 1, while those which are free are counted with weight Q, it follows that

$$\begin{aligned} Z_{ff} &= \langle Q^{N_c+N_L+N_R+N_0}\rangle \qquad Z_{aa} = \langle Q^{N_0}\rangle && (10)\\ Z_{af} &= \langle Q^{N_R+N_0}\rangle \qquad\qquad Z_{fa} = \langle Q^{N_L+N_0}\rangle && (11)\end{aligned}$$

where f now denotes that the boundary spins are free on that portion of the boundary (they are always free on the complement of $\gamma_1\cup\gamma_2$.) Then straightforward algebra shows that

$$\langle N_c\rangle = (\partial/\partial Q)|_{Q=1}(Z_{ff}+Z_{aa}-Z_{fa}-Z_{af}) = (\partial/\partial Q)|_{Q=1}(Z_{ff}Z_{aa}/Z_{fa}Z_{af}) \tag{12}$$

where the last equality holds because all the partition functions equal 1 at $Q=1$.

[6]Note that the crossing probability in the random cluster model with $Q\neq 1$ is not given by the (normalised) difference $(Z_{aa}-Z_{ab})/Z_{aa}$, since this counts clusters which touch γ_1 and/or γ_2 with weight 1, rather than Q.

[7]We can think of the crossing probability as $P(N_c\geq 1)$, where N_c is the number of crossing clusters. It is also possible to design more complicated boundary conditions which give $P(N_c\geq n)$ for any integer $n\geq 1$. See [9].

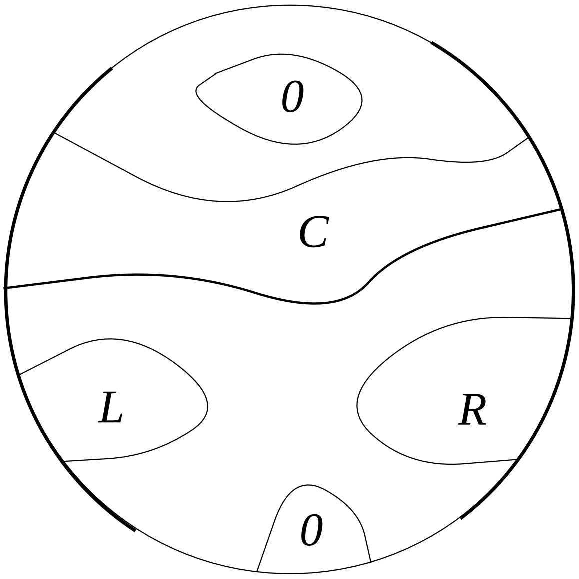

Figure 3: Different types of cluster, which touch both, one or none of γ_1 and γ_2.

It will be the aim in the rest of these lectures to derive explicit expressions for these partition functions, in the continuum limit, and for Q close to 1.

3 The continuum limit of critical lattice models.

The Potts model in two dimensions is known to have a critical point, with a divergent correlation length, for $0 \leq Q \leq 4$. We now move onto less solid ground. What I am now going to assert is based on evidence from the exactly solved Ising model $Q = 2$, as well as the analysis of renormalised perturbation theory near the upper critical dimension.[8]

At the critical point, these theories are believed to be *scale invariant.* What this means for correlation functions is the following. Consider for example, the correlations $\langle s(r_1)s(r_2)\ldots s(r_n)\rangle$ of the local lattice magnetisation in the Ising model. Since $s(r)=\pm 1$, this quantity has no dimensions. At the critical point, in the limit where the lattice spacing $a\to 0$ with the points r_j kept fixed, it behaves like a^{nx} times a function of $r_1,\ldots,r_n$, where x is a pure number (equal to $\frac{1}{8}$ for the Ising model magnetisation.) This means that we can define a *scaling operator* $\phi(r)\equiv a^{-x}s(r)$ such that the limit $a\to 0$ of the correlation functions $\langle\phi(r_1)\phi(r_2)\ldots\phi(r_n)\rangle$ exists. The pure number x, which we should denote as x_ϕ, is called the *scaling dimension* of ϕ.

Moreover, as long as the original lattice model has sufficient symmetry under finite rotations, the limit is invariant under infinitesimal rotations. Thus, far away from any boundaries, the two-point correlation function $\langle\phi(r_1)\phi(r_2)\rangle$ depends on the separation $|r_1-r_2|$ only, and, on dimensional grounds, must therefore have the form[8]

$$\langle\phi(r_1)\phi(r_2)\rangle=\text{const.}\,|r_1-r_2|^{-2x_\phi} \tag{13}$$

The above statement is strictly true only when the points r_j are distinct, in the limit $a\to 0$. That is, they are separated by an infinite number of lattice spacings. When their relative distances are kept fixed in units of a, other scaling operators may arise. For example, the product $s(r_1')s(r_1'')$ on neighbouring sites may be thought of as being proportional to the local energy density in the Ising model. Correlation functions of this quantity behave in a similar way to those of the magnetisation, but they define a different scaling operator, with a different scaling dimension. In general there is an infinite number of scaling operators $\phi_k(r)$, each with their own scaling dimensions x_k. Arbitrary local products S_j of spins which are separated by distances $O(a)$, are given asymptotically as linear combinations of these scaling operators:

$$S_j(r)=\sum_k A_{jk}a^{x_k}\phi_k(r) \tag{14}$$

where the dimensionless coefficients A_{jk} are of order unity, and correlations of the scaling operators all have a limit as $a\to 0$. These are rotationally

[8]In the case of the magnetisation, $2x_\phi$ is conventionally denoted by $(d-2+\eta)$.

and translationally invariant (far from a boundary), and also *scale covariant*: under a scale transformation $r \to r' \equiv b^{-1} r$

$$\langle \phi_1(r_1')\phi_2(r_2')\ldots\phi_n(r_n')\rangle = b^{\sum_k x_k}\langle \phi_1(r_1)\phi_2(r_2)\ldots\phi_n(r_n)\rangle \tag{15}$$

In (14) the coefficients depend on the particular lattice and so on, but the scaling dimensions x_ϕ are *universal*. For example, in the Potts model they depend only on Q and the dimensionality of the system, assuming that the interactions are short-ranged.[9]

3.1 Boundary operators.

The above statements about the existence of the continuum limit remain valid in the presence of boundaries. Of course the boundary now breaks global rotational and translational invariance. Taking for simplicity the boundary to run along the real axis, so that the system occupies the upper half-plane $y > 0$, then the two-point function of a scaling operator has the general form[10]

$$\langle \phi(x_1,y_1)\phi(x_2,y_2)\rangle = (y_1y_2)^{-x_\phi} F\left(|x_1-x_2|/y_1, |x_1-x_2|/y_2\right) \tag{16}$$

where F is a scaling function. This is valid for all y_1 and y_2 strictly positive, in the limit $a \to 0$. F behaves in such a way as to recover the bulk result $((x_1-x_2)^2+(y_1-y_2)^2)^{-x_\phi}$ as both y_1, $y_2 \to \infty$, but it turns out that the limit

[9] One of the important properties of the scaling operators is that they are complete in the following sense. If, in the above correlation function, we take the separation $|r_1-r_2|$ to be much smaller than the distances $|r_1-r_k|$ to the other points with $k \geq 3$, this correlation function may be expressed as a sum over correlation functions with only a single operator at the mid-point, say, of r_1 and r_2: $\langle\phi_1(r_1)\phi_2(r_2)\ldots\rangle = \sum_k c_{12k}|r_1-r_2|^{x_k-x_1-x_2}\langle\phi_k((r_1+r_2)/2)\ldots\rangle$ where the dependence on r_1-r_2 is dictated by scale covariance and rotational invariance (we have assumed that all the operators are scalar under rotations, which is a slight over-simplification). The important thing is that the coefficients c_{ijk} are also universal, and in particular independent of the other operators hidden in the $\ldots$. We can thus remove the $\langle\ldots\rangle$. The result is called the *operator product expansion (OPE)*.

as y_1 and/or $y_2 \to 0$ is singular. In that limit we find instead the behaviour

$$\langle \phi(x_1, y_1)\phi(x_2, y_2)\rangle \sim \sum_k B_k^2 (y_1 y_2)^{-x_\phi + \tilde{x}_k} |x_1 - x_2|^{-2\tilde{x}_k} \tag{17}$$

where the B_k are (universal) constants and the $\tilde{x}_k$ are a set of *boundary scaling dimensions*. Thus there is a separate set of scaling operators $\tilde{\phi}_k$ defined on the boundary $y = 0$, which have scaling dimensions $\tilde{x}_k$ which are in general different from those in the bulk.[10] It turns out that the set of such operators depends not only on the bulk universality class, but also on the nature of the boundary condition, for example whether the lattice spins are free or fixed.

3.2 Boundary condition changing operators.

Once the idea of boundary scaling operators was understood, it was realised[11] that, at least in two dimensions when the boundary is one-dimensional, there is another set of objects which should possess similar properties under scale transformations. These are points on the boundary where the boundary condition changes from one type to another. Let us illustrate this with a relevant example from the Potts model. Consider a Potts model in the half-disc $r < R$, $y \geq 0$, see Fig. 4. Compare the case when (a) the boundary conditions on the Potts spins are free (f) at every point of the boundary except for the interval (z_1, z_2) of the real axis, where they are fixed to state a, to the case (b) when they are free everywhere on the boundary. Denote the corresponding partition functions by Z_{af} and Z_f. Then the ratio Z_{af}/Z_f has a finite limit as $R \to \infty$, [11] which therefore depends only on $|z_1 - z_2|$. It factorises into two parts: the leading, non-universal, term is $\sim \exp(-(f_a^s - f_f^s)(z_2 - z_1))$, where f_α^s is the *boundary free energy* per unit length, corresponding to boundary condition α; and a residual universal factor, which decays with a power law

[10] (17) is equivalent to the *bulk-boundary* expansion $\phi(x, y) = \sum_k B_k y^{-x_\phi + \tilde{x}_k} \tilde{\phi}_k(x)$.

[11] This is not obvious, at criticality, and needs to be proved.

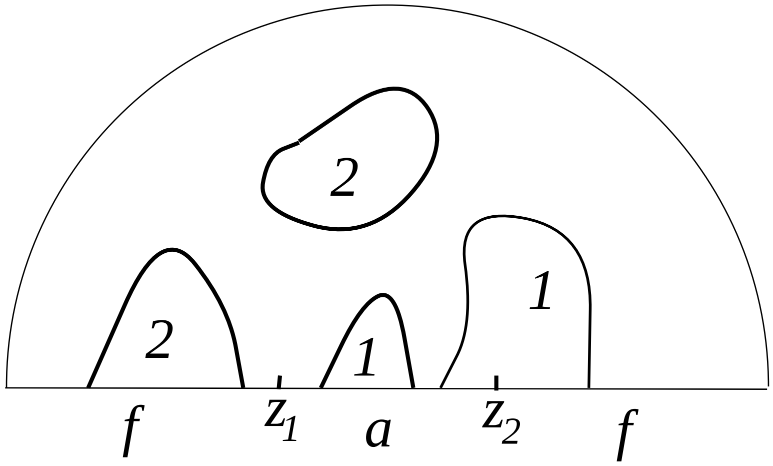

Figure 4: Potts model in the half-plane. The two-point function of the bcc operator counts clusters which touch the real axis in the interval (z_1, z_2).

at criticality. In what follows, we shall ignore the non-universal term: it cancels in most important quantities.

We may think of this ratio as defining the two-point correlation function of *boundary condition changing operators* (bcc operators) with a well-defined scaling dimension:

$$\langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\rangle \equiv \lim_{a\to 0} a^{-2x_\phi}(Z_{af}/Z_f) \propto |z_1 - z_2|^{-2x_\phi} \tag{18}$$

Within the Potts model, we expect the scaling dimension of this bcc operator to be universal, depending only on Q. More complicated sets of boundary conditions may be considered. For example, the partition function Z_{ab} in (9) which we shall need to compute the crossing probability is related to the correlation function $\langle \phi_{f|a}\phi_{a|f}\phi_{f|b}\phi_{b|f}\rangle$.

3.3 Finite-size scaling of the transfer matrix.

Consider now a lattice model defined on an infinitely long strip of width L, parametrised by the coordinates (u, v) with $0 \leq v \leq L$. We could consider periodic boundary conditions in the v-direction, but for our purposes it will be more useful to think about particular boundary conditions, labelled say by α and β (which do not have to be the same) on either edge of the strip. A convenient way of discussing this geometry is through the transfer matrix[12] $\hat{T}_{\alpha\beta}(L)$, which is a finite matrix whose rows and columns are labelled respectively by the states of the spins in neighbouring rows $(u, u+a)$, and whose elements are the Boltzmann weights for the two rows. If we take the strip to have finite length W in the u-direction, with periodic boundary conditions in that direction, the partition function is given by

$$Z = \mathrm{Tr}\,\hat{T}^{W/a} = \sum_n \Lambda_n^{W/a} \tag{19}$$

where the Λ_n are a complete set of eigenvalues of $\hat{T}$ (which of course depend on α, β and L.) In this picture the lattice spins $s(u,v)$ themselves become matrices $\hat{s}(v)$ (diagonal in this basis), acting on the same space as does $\hat{T}$. If the eigenstate corresponding to the largest eigenvalue of $\hat{T}$ is denoted by $|0\rangle$, then a two-point correlation function may be written

$$\langle s(u_2,v_2)s(u_1,v_1)\rangle = \langle 0|\hat{s}(v_2)(\hat{T}/\Lambda_0)^{(u_2-u_1)/a}\hat{s}(v_1)|0\rangle \tag{20}$$

and so, by inserting a complete set of eigenstates of $\hat{T}$, we see that it decays along the strip as a sum of terms of the form $(\Lambda_n/\Lambda_0)^{(u_2-u_1)/a}$ such that $\langle n|\hat{s}|0\rangle \neq 0$. Similar equations hold for the correlation function of any local product $\hat{S}_j(v)$ of lattice spins.

How do these quantities behave in the continuum limit $a \to 0$, with L fixed? The space of states becomes infinite-dimensional, but we shall not worry too much about its precise structure. The scaling 'operators' $\phi(u,v)$ become true local operators $\hat{\phi}(v)$ acting on this space. It is useful to write

[12] I shall attempt to be consistent and denote true operators by a hat ($\hat{}$). Note that the scaling 'operators' introduced earlier are not true operators, but simply local densities, which commute with each other.

the transfer matrix itself as

$$\hat{T}_{\alpha\beta}(L) = \exp(-a\hat{H}_{\alpha\beta}(L)) \tag{21}$$

where $\hat{H}$ has eigenvalues E_n, so that correlation functions decay as a sum of terms $e^{-(E_n-E_0)(u_2-u_1)}$. The existence of the continuum limit implies that the gaps $E_n - E_0$ have a limit as $a \to 0$. Scaling then implies that, at the critical point, each of these gaps must be proportional to L^{-1}. As we shall soon see, conformal invariance relates the constants of proportionality to scaling dimensions of operators.

4 From scale invariance to conformal invariance.

Let us recall the equation (15) for the behaviour of a general correlation function under a scale transformation $r \to r' \equiv b^{-1}r$:

$$\langle\phi_1(r_1')\phi_2(r_2')\ldots\rangle = b^{\sum_k x_k}\langle\phi_1(r_1)\phi_2(r_2)\ldots\rangle \tag{22}$$

One way to understand this equation is from the existence of the continuum limit $a \to 0$, and the absence of any other length scale at the critical point: if this limit does exist, it should not matter whether we rescale the lattice spacing in the underlying lattice model by a factor b: $a \to ba$, in the sense that the measure of those degrees of freedom which survive the continuum limit should be unchanged. But we can think of this rescaling (which is a kind of renormalisation group transformation) as keeping the lattice spacing fixed, and rescaling $r \to r' \equiv b^{-1}r$.

Now we make the following bold generalisation: since the measure is ultimately a product over local Boltzmann weights, we may equally well make a rescaling $a \to b(r)a$ where the factor b now varies smoothly with r (on a scale $\gg a$.) So we generalise (22) to

$$\langle\phi_1(r_1')\phi_2(r_2')\ldots\rangle = \prod_k b(r_k)^{x_k}\langle\phi_1(r_1)\phi_2(r_2)\ldots\rangle \tag{23}$$

where $b(r) = |\partial r/\partial r'|$, the jacobian of the transformation.

What are the allowed transformations $r \to r'$? Locally, they must correspond to a mixture of a scale transformation together with a possible rotation and translation. Such transformations are called *conformal.*[13] An example is shown in Fig. 5.[14] So (23) describes the *conformal covariance* of an arbitrary correlation function.[15] Once again, it is not rigorously founded, but rather abstracted from exactly solved cases like the Ising model, and from the structure of renormalised perturbation theory. It is generally believed that the continuum limit of any model which is invariant under scale transformations, rotations and translations, and has short-ranged interactions, is also conformally invariant. We are going to assume that it is valid for the continuum limit of the critical Potts model with $0 \leq Q \leq 4$.

In two dimensions, conformal invariance is particularly powerful, because if we label the points of the plane by a complex number $z = x + iy$, any analytic function $z \to z' = f(z)$ defines a conformal transformation, at least at points where $f'(z) \neq 0$.

4.1 Some simple consequences.

Let us consider for definiteness correlations of boundary operators, since it is these we need for our problem. They may be ordinary limits of bulk operators at the boundary, or they may be bcc operators. Just as scale covariance fixes the form of the two-point functions, conformal invariance gives further

[13]Strictly speaking, conformal transformations operate on the metric rather than the coordinates, but this distinction is not important here.

[14]It is not possible to have a non-trivial conformal transformation without some element of local rotation as well. This means that anisotropic critical points, which occur, for example, in directed percolation or at a Lifshitz point, are not conformally invariant.

[15]This is written for the case of operators which transform as scalars under rotations. A deeper analysis also shows that (23) cannot be true for all scaling operators: for example it is easy to show that if it is true for ϕ it cannot in general be true for $\nabla^2\phi$. However, this analysis shows that it is true for a subset of operators called *primary*, and the transformation laws for all other operators may be derived from these.

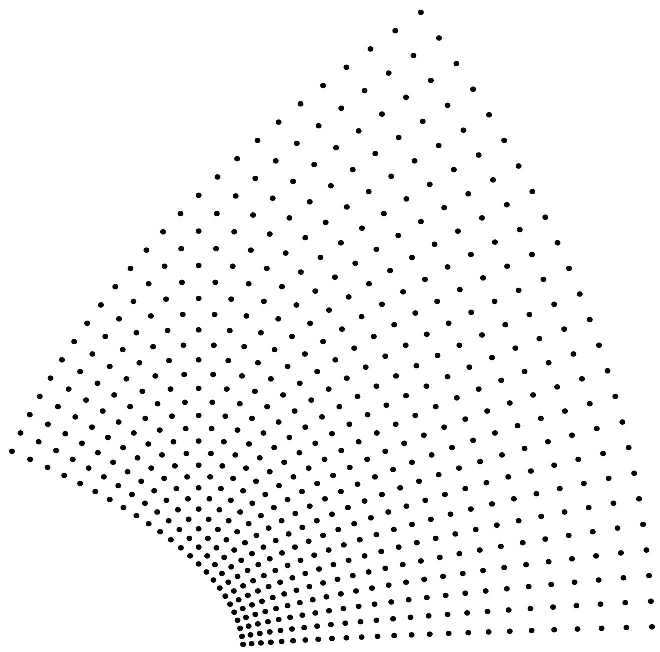

Figure 5: Example of a conformal transformation, locally equivalent to a scale transformation plus a rotation.

information about the higher-point functions. If we consider the upper half plane, then the set of conformal transformations which preserve this are the real Möbius transformations

$$z \to z' = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0, \tag{24}$$

where a, b, c and d are real (this is to preserve the real axis.) Given a 4-point function $\langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle$, with the z_j real, it is therefore possible in general to choose the parameters in (24) such that z_2, z_3 and z_4 are mapped into three pre-assigned points, say, $(0, \infty, 1)$, corresponding to $z' = (z - z_2)(z_4 - z_3)/(z - z_3)(z_4 - z_2)$. However there is always an invariant of such a transformation: it is the cross-ratio (anharmonic ratio)

$$\eta \equiv (z_{12}z_{34}/z_{13}z_{24}) \tag{25}$$

(where the notation $z_{kl} \equiv z_k - z_l$ has been introduced) so that, in this example, z_1 gets mapped into η. We may now apply the transformation formula (23). The correlation function on the left is some (as yet) unknown function of η, and it is simply a matter of working out the jacobian. After some algebra,[16]

$$\langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle = \left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}}\right)^{2x_\phi} F(\eta) \tag{26}$$

4.2 Application to crossing probabilities.

We are already in a position where we can deduce part of the main result. We begin with the partition functions corresponding to the geometry of Fig. 1, in which the Potts spins on the segments γ_1 and γ_2 are fixed, and the rest are free. As we showed earlier, the crossing probability is given by the limit

[16]This is the simpler case when all four operators have the same scaling dimension. A more general result also holds.

as $Q \to 1$ of the partition functions Z_{aa} and Z_{ab}. According to our analysis above these may be written, as $a \to 0$,

$$Z_{aa}/Z_f \sim a^{4x(Q)} \langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\phi_{f|a}(z_3)\phi_{a|f}(z_4)\rangle \tag{27}$$

$$Z_{ab}/Z_f \sim a^{4x(Q)} \langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\phi_{f|b}(z_3)\phi_{b|f}(z_4)\rangle \tag{28}$$

Here we have introduced the scaling dimension $x(Q)$ of the (free|fixed) bcc operator, which will depend on Q, but not on a or b. Now the Riemann mapping theorem assures us that there exists a conformal mapping of the interior of the region $\mathcal{R}$ to that of the unit disc, and thence, by a simple Möbius transformation, to the upper half plane $y > 0$. Thus we may use (23) to relate the ratios of partition functions in the two geometries.[17] However, this will bring in non-trivial factors from the jacobian, evaluated at the points z_j, raised to the power $x(Q)$, so that the ratio of partition functions is *not*, in general, conformally invariant.

However, a remarkable thing happens at $Q = 1$. Consider the two-point function $\langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\rangle$ in the upper half plane geometry. As discussed earlier, this is proportional to the ratio of correlation functions Z_a/Z_f, and, at the critical point, decays as $|z_1 - z_2|^{-2x(Q)}$. In the random cluster model, Z_a counts all clusters which touch the real axis in the interval (z_1, z_2) with a weight 1, while counting all other clusters with weight Q (see Fig. 4.) Denoting the total number of each type of cluster by N_1 and N_2 respectively, we have $Z_a = \langle Q^{N_2}\rangle$ and $Z_f = \langle Q^{N_1+N_2}\rangle$. Trivially, then, both partition functions equal one at $Q = 1$, so that their ratio is independent of z_1 and z_2. Thus

$$x(1) = 0 \tag{29}$$

[17] In doing this we must assume that the transformation is conformal also at the points z_j. This requires that the boundary Γ be differentiable at that point. There are interesting additional factors when the boundary has a corner at one of these points (as happens in the relevant case of the rectangle). However all these factors are raised to a power proportional to $x(Q)$, so are not important at $Q = 1$. See Ref. [13].

an innocent but remarkably interesting equation from the point of view of conformal field theory.[18] From our perspective it means that the crossing probabilities in percolation (at least those which may be expressed in terms of $\phi_{f|a}$) are conformally *invariant*, in the sense stated in the Introduction. Thus percolation has stronger properties under conformal transformations than do more general critical theories. This may be traced to the fact that its partition function is equal to 1.

5 Scaling operators and states in conformal field theory.

We have already discussed the continuum limit of the transfer matrix on strip of width L, and introduced the concept of a state space spanned by the eigenvectors of the operator $\hat{H}$. Now we shall see that this has a very simple structure due to conformal invariance. Consider the upper half plane, with boundary conditions labelled by α along the real axis. Consider the conformal mapping

$$z = x + iy \to w = u + iv = (L/\pi) \ln z \tag{30}$$

which maps the upper half plane into such a strip, with boundary conditions α on both edges. If we apply the transformation law (23) to the correlation function $\langle\tilde{\phi}(z_1)\tilde{\phi}(z_2)\rangle$ in the half-plane (where $\tilde{\phi}$ is some boundary operator and we assume that $0 < z_1 < z_2$), we find

$$\langle\tilde{\phi}(u_1, 0)\tilde{\phi}(u_2, 0)\rangle = \left[\frac{\pi/L}{\sinh((\pi/L)|u_1 - u_2|)}\right]^{2x_{\tilde{\phi}}} \tag{31}$$

[18]Note that if we ask for the mean number $\langle N_1\rangle$ of clusters touching the interval (z_1, z_2) we must first differentiate with respect to Q. The result is a non-universal term, linear in $|z_1 - z_2|$, coming from the difference in boundary free energies per unit length between free and fixed boundary conditions (or, more physically, from the contribution of small clusters), plus a universal term $-2x'(1)\ln|z_1 - z_2|$. As we show later, this prefactor is $\sqrt{3}/4\pi$.

For $|u_1 - u_2| \ll L$ this reduces to the same result as in the half-plane, but in the opposite limit it decays as $\exp(-(\pi\tilde{x}_\phi/L)|u_1 - u_2|)$. This means that there must be an eigenstate of $\hat{H}$ with a gap $E_n - E_0 = \pi\tilde{x}_\phi/L$. In fact it may be argued that to each boundary scaling operator $\tilde{\phi}$ there is an eigenstate. This may be identified by taking $u_1 \to -\infty$ (so that $x_1 \to 0$), in which case only the state $\langle n|$ with the lowest value of E_n such that $\langle n|\hat{\tilde{\phi}}|0\rangle \neq 0$ will contribute. A less obvious statement is that to each eigenstate there is an operator. This can be seen by choosing the strip to be in some particular eigenstate $|n\rangle$ at $u = \pm\infty$ and computing correlation functions on the strip with this state rather than the vacuum $|0\rangle$. On transforming these back to the half-plane, they turn out to be exactly as if we had an operator at the origin, with scaling dimension $(L/\pi)(E_n - E_0)$. Thus we have one of the most important results of two-dimensional conformal field theory:[19]

- There is a (1-1) correspondence between the (boundary) scaling operators and the eigenstates of $\hat{H}$ in the strip.

In what follows, we shall normalise so that $E_0(L) = 0$.[20] This means that a boundary scaling operator $\tilde{\phi}$, inserted near the origin, corresponds to an eigenstate with $E = \pi x_\phi/L$ in the strip. We may extent this to bcc operators: if we now consider a strip with different boundary conditions α and β on either edge, this corresponds to an insertion of the bcc operator $\phi_{\alpha|\beta}$, and so the the lowest eigenvalue of $\hat{H}_{\alpha\beta}$ is $\pi x_{\phi_{\alpha|\beta}}/L$.

[19] A more field-theoretic way to state this is that if $\hat{H}$ generates translations in u along the strip, then $(L/\pi)(\hat{H} - E_0)$ generates scale transformations in the half-plane. Eigenstates of the latter correspond to scaling operators.

[20] It is possible to show that E_0 is in general independent of α (in CFTs in which all the scaling dimensions are non-negative), so this is consistent. However, there is useful information in the L-dependence: if we normalise so that $E_0(\infty) = 0$, then $E_0(L) = -\pi c/24L$, where c is the central charge of the CFT.

5.1 Descendent operators.

Let us consider in more detail the structure of the set of scaling operators. It is useful first to think about infinitesimal conformal transformations (which preserve the upper half-plane). The simplest ones are the translations, $z \to z + a_{-1}$ (where a_{-1} is an infinitesimal parameter), under which a scaling operator $\tilde{\phi}(0)$ transforms according to

$$\tilde{\phi}(0) \to \tilde{\phi}(a_{-1}) = \tilde{\phi}(0) + a_{-1}[L_{-1}\tilde{\phi}](0) \tag{32}$$

where $L_{-1}\tilde{\phi}$ is nothing but $(\partial/\partial z)\tilde{\phi}$. There are also the scale transformations $z \to z + a_0 z$, under which

$$\tilde{\phi}(0) \to (1 + a_0)^{x_{\tilde{\phi}}}\tilde{\phi}(0) \sim \tilde{\phi}(0) + a_0[L_0\tilde{\phi}](0) \tag{33}$$

where $L_0\tilde{\phi} = x_{\tilde{\phi}}\tilde{\phi}$. The statement that the correlations of $\tilde{\phi}$ satisfy (23) is equivalent to assuming that for a general infinitesimal conformal transformation $z \to z + \sum_{n=-1}^{\infty} a_n z^{n+1}$ which is regular at the origin,

$$\tilde{\phi}(0) \to \tilde{\phi}(0) + a_{-1}[L_{-1}\tilde{\phi}](0) + a_0[L_0\tilde{\phi}](0), \tag{34}$$

with no further terms.

However, a more general conformal transformation of this kind is bound to have singularities elsewhere in the complex plane (including infinity). In order to understand the implication of this, let us consider a more general infinitesimal transformation in the form of a Laurent series, which may be singular at the origin: $z \to z + \sum_{n=-\infty}^{\infty} a_n z^{n+1}$. No matter how small we choose the parameters a_n for $n < -1$, for sufficiently small z this transformation blows up. We deal with this by imagining cutting out a small half-disc of fixed radius around the origin, and, inside this, smoothing out the transformation. By definition it cannot be conformal in this region. The effect of this from the point of view of the correlation functions with other operators far from the origin is to generate *new* operators, called the *descendents* $\tilde{\phi}$. Thus in general we have

$$\tilde{\phi}(0) \to \tilde{\phi}(0) + \sum_{n=-\infty}^{\infty} a_n [L_n \tilde{\phi}](0) \tag{35}$$

Notice that by dimensional analysis the descendent operators $L_n \tilde{\phi}$ have scaling dimensions $x_{\tilde{\phi}} + (-n)$.

We can keep doing this, by examining the properties of these descendent operators under further infinitesimal conformal transformations. We thus generate an enormous set of descendent operators $L_{n_1} L_{n_2} \ldots \tilde{\phi}$, with scaling dimensions $x_{\tilde{\phi}} + \sum_j (-n_j)$. This, in turn, implies that to each simple eigenstate of the strip transfer matrix $\hat{H}$ corresonds a whole tower of other eigenstates, with eigenvalues spaced by integer multiples of π/L.[21]

6 Null operators and differential equations.

In general, with each scaling operator $\tilde{\phi}$ is associated an infinite number of descendents. However, they do not all have to be independent, and in many important applications, they are not. In that case, as we shall show, the correlation functions of $\tilde{\phi}$ satisfy simple linear differential equations.

Let us consider the case of interest, the bcc operator $\phi_{f|a}$, with scaling dimension $x(Q)$. Consider the partition function Z_{fa} of an annulus $0 \leq v \leq L$, $0 \leq u < W$, with periodic boundary conditions in the u-direction, free

[21] These descendent operators do not necessarily satisfy the condition that $[L_n \phi] = 0$ for $n \geq 1$, that is they are not primary and they do not satisfy (23) without adding some additional terms. In general, however, they are related to other descendent operators, with lower scaling dimension, are already constructed. For example, $[L_1 L_{-1} \phi] = 2[L_0 \phi]$. This algebraic structure is realised by thinking of the $\hat{L}_n$ as operators which act on the states of the transfer matrix. They are in fact the Fourier components $\hat{L}_n = \int_0^L e^{\pi i n v/L} \hat{T}(v) dv$ of a special operator called the stress-energy tensor. They satisfy the Virasoro algebra $[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n,-m}$, which plays a very important role in the algebraic development of conformal field theory. However, we do not really need this formalism for the purposes of these lectures.

boundary conditions f on $v=0$ and fixed boundary conditions a on $v=L$. As discussed above, we may write

$$Z_{fa}(Q)=\mathrm{Tr}\,\exp(-W\hat{H}_{fa}(L))=\sum_j q^{x_j} \tag{36}$$

where $q\equiv e^{-\pi W/L}$ and the sum is over all scaling operators, and their descendents, which may occur consistent with the given boundary conditions. The lowest is $x_0=x(Q)$, corresponding to the operator $\phi_{f|a}$ itself. Let us examine the first few eigenstates above this, bearing in mind the fact that, as $Q\to 1$, $Z_{fa}(Q)\to 1$. The first consequence is that, as already noted $\lim_{Q\to 1}x(Q)=0$. At $Q=1$ this gives a contribution 1 to Z. All other contributions must therefore cancel among themselves in this limit. There is a (unique) operator $L_{-1}\phi_{f|a}$, with scaling dimension $x(Q)+1$, which gives a contribution q to Z at $Q=1$, which must be cancelled. The only way is by other primary operator(s) $\tilde{\psi}$ whose scaling dimensions $x_\psi(Q)\to 1$ as $Q\to 1$. In fact, there are candidates for such operators: close to the bcc operator $\phi_{f|a}$ the S_Q permutation symmetry of the Potts model is broken down to S_{Q-1}. The lattice operators $\delta_{s(r),b}$ with $b\neq a$ thus span a $(Q-2)$-dimensional space. We identify $\tilde{\psi}$ with the continuum limit of these operators, with degeneracy $(Q-2)$. Thus

$$Z_{f|a}(Q)=q^{x(Q)}+q^{x(Q)+1}+(Q-2)q^{x_\psi(Q)}+\cdots \tag{37}$$

So far, so good.[22] However, at the next level we have in principle the operators $L_{-2}\phi_{f|a}$, $L_{-1}^2\phi_{f|a}$ (each with weight 1), and $L_{-1}\psi$, with weight $(Q-2)$. Together these would give a contribution q^2 to Z_{af}. There are two possible resolutions: either there must be yet another new primary operator ψ' with dimension $x_{\psi'}(Q)\to 2$ as $Q\to 1$ and degeneracy $-1+O(Q-1)$; or the

[22]Note that we are ignoring the possible contributions of operators with non-integer scaling dimensions at $Q=1$. These are believed not to arise, but in any case would not affect the argument.

two operators $L_{-2}\phi_{f|a}$ and $L_{-1}^2\phi_{f|a}$ are not independent. In that case the combination $L_{-2}\phi_{f|a} - \kappa L_{-1}^2\phi_{f|a}$ (for some number κ) corresponds to a *null state*, which does not contribute to the partition function. We shall *assume* that the latter solution is chosen.[23]

Let us now go back to the half-plane and examine the consequences of this for the four-point function

$$G(z_1, z_2, z_3, z_4) \equiv \langle \phi_{f|a}(z_1)\phi_{a|f}(z_2)\phi_{f|a}(z_3)\phi_{a|f}(z_4)\rangle, \tag{38}$$

where the points z_j lie on the real axis. Consider the infinitesimal conformal transformation

$$z \to z' = z + a(z) = z + a_{-2}(z - z_1)^{-1} \tag{39}$$

At the points z_j $(2 \leq j \leq 4)$ this is regular, and the corresponding operators transform simply according to (34). At z_1 however it is singular, and must be regularised in the manner described earlier. This then generates the new operator $a_{-2}[L_{-2}\phi_{f|a}](z_1)$. We may now equate this to $\kappa[L_{-1}^2\phi_{f|a}](z_1) = \kappa(\partial/\partial z_1)^2\phi_{f|a}](z_1)$, where κ is an (as yet) unknown constant. Setting the total variation with respect to a_{-2} to zero, we therefore find that G satisfies the differential equation

$$\left[\kappa\frac{\partial^2}{\partial z_1^2} + \sum_{j=2}^{4}\left(\frac{1}{z_j - z_1}\frac{\partial}{\partial z_j} - \frac{x(Q)}{(z_j - z_1)^2}\right)\right] G(z_1, z_2, z_3, z_4) = 0 \tag{40}$$

This is a linear partial differential equation. However, if we substitute in the form (26) (which, recall, followed from the covariance under Möbius

[23] This is the weakest point in the whole argument. However, there is further supporting evidence: at $Q = 2$ and at $Q = 3$ the same property is known to hold[14]. In addition, the operator product expansion of two such operators is known, by the so-called *fusion rules*[5], to generate another operator degenerate at level 3: this has been identified much earlier[14] with the magnetisation operator at a free boundary, which, by duality, has the same scaling dimension as the bcc operator $\phi_{a|b}$.

transformations), we find an ordinary second-order linear differential equation for $F(\eta)$ of a type very familiar to mathematical physicists. In fact, it is a hypergeometric equation, with regular singular points at $(0,\infty,1)$. A lot of tedious algebra can be avoided by identifying the indices of this equation, that is, the possible power law behaviour of the general solution near the singular points. For example, if $F(\eta)\sim\eta^\alpha$ as $\eta\to 0$, this implies that $G\sim(z_{12}z_{34})^{\alpha-2x(Q)}(z_{13}z_{24})^{-\alpha}$ in the limit where $z_{12}z_{34}\ll z_{13}z_{24}$. Substituting into (40) and equating the coefficient of $z_{12}^{\alpha-x(Q)-2}$ to zero gives

$$\kappa(\alpha-2x)(\alpha-2x-1)+(\alpha-2x)-x=0 \tag{41}$$

Now we know that a particular solution is G, which in this limit should factorise:

$$G\sim\langle\phi_{f|a}(z_1)\phi_{a|f}(z_2)\rangle\langle\phi_{f|a}(z_3)\phi_{a|f}(z_4)\rangle\sim z_{12}^{-2x(Q)}z_{34}^{-2x(Q)} \tag{42}$$

This is because, in that limit, clusters which touch both segments (z_1,z_2) and (z_3,z_4) are very rare. Therefore the indicial equation (41) must have a solution with $\alpha=0$. This implies that

$$\kappa=\frac{3}{2(2x(Q)+1)} \tag{43}$$

The other solution is then $\alpha=\frac{1}{3}+\frac{8}{3}x(Q)$.

A similar analysis may be made of the indicial equations at the other singular points. At $\eta=1$ we also find $(0,\frac{1}{3}+\frac{8}{3}x(Q))$, and at ∞, $(-4x(Q),\frac{1}{3}-\frac{4}{3}x(Q))$.[24]

The standard form of the hypergeometric equation is

$$\eta(1-\eta)F''+[c-(a+b+1)\eta]F'-abF=0 \tag{44}$$

[24] The indices at $\eta=\infty$ are conventionally defined by $F\sim\eta^{-\alpha}$. There is a result that the 6 indices of such an equation must sum to 1. This explains the appearance of the fraction $\frac{1}{3}$ when $x(Q)=0$.

for which the indices are $(0,1-c)$, (a,b) and $(0,c-a-b)$ at $(1,\infty,0)$ respectively. This we have in our case

$$a=-4x(Q), \qquad b=\tfrac{1}{3}-\tfrac{4}{3}x(Q), \qquad c=\tfrac{2}{3}-\tfrac{8}{3}x(Q) \tag{45}$$

This equation has two independent solutions. The one which is regular at $\eta=0$ is proportional to

$${}_2F_1(a,b;c;\eta)=1+\frac{ab}{c}\frac{\eta}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{\eta^2}{2!}+\cdots \tag{46}$$

which converges for $|\eta|<1$. There are in fact many different ways of writing down two independent solutions: a particularly useful pair are

$$\begin{aligned}&{}_2F_1(a,b;a+b+1-c;1-\eta)\\ &={}_2F_1(-4x,\tfrac{1}{3}-\tfrac{4}{3}x;\tfrac{2}{3}-\tfrac{8}{3}x;1-\eta) \qquad \text{and}\end{aligned} \tag{47}$$

$$\begin{aligned}&(1-\eta)^{c-a-b}{}_2F_1(c-b,c-a;c-a-b+1;1-\eta)\\ &=(1-\eta)^{\frac{1}{3}+\frac{8}{3}x}{}_2F_1(\tfrac{1}{3}-\tfrac{4}{3}x,\tfrac{2}{3}+\tfrac{4}{3}x;\tfrac{4}{3}+\tfrac{8}{3}x;1-\eta)\end{aligned} \tag{48}$$

Although we have been considering the four-point function of bcc operators corresponding to Z_{aa}, the differential equation should hold equally well for that corresponding to Z_{ab}. In fact, the two independent solutions above are (apart from constants) just the solutions F_{aa} and F_{ab} corresponding to Z_{aa} and Z_{ab} respectively. This is because $\eta\to 1$ corresponds to $z_2\to z_3$. In the second case (ab), this introduces the non-trivial bcc operator $\phi_{a|b}$ at z_2, so cannot contain a term $\propto(1-\eta)^0$ (incidentally, we see that this operator must have dimension $\frac{1}{3}+\frac{8}{3}x(Q)$); while in the first case, the opposite is true. The coefficients may then found from the requirement that, as $\eta\to 0$, $F_{aa}\sim F_{ab}\to 1$, and the identity

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{49}$$

We therefore have the final results for the partition functions in the half-plane (normalised so that $Z_f = 1$)

$$\begin{aligned} Z_{aa} &= \zeta^{2x(Q)} \frac{\Gamma(\frac{2}{3}+\frac{4}{3}x)\Gamma(\frac{1}{3}-\frac{4}{3}x)}{\Gamma(\frac{2}{3}-\frac{8}{3}x)\Gamma(\frac{1}{3}+\frac{8}{3}x)} {}_2F_1(-4x, \tfrac{1}{3}-\tfrac{4}{3}x; \tfrac{2}{3}-\tfrac{8}{3}x; 1-\eta) \quad (50) \\ Z_{ab} &= \zeta^{2x(Q)} \frac{\Gamma(1+x)\Gamma(\frac{2}{3}+\frac{4}{3}x)}{\Gamma(\frac{4}{3}+\frac{8}{3}x)\Gamma(\frac{1}{3}+\frac{8}{3}x)} (1-\eta)^{\frac{1}{3}+\frac{8}{3}x} \\ &\quad \times {}_2F_1(\tfrac{1}{3}-\tfrac{4}{3}x, \tfrac{2}{3}+\tfrac{4}{3}x; \tfrac{4}{3}+\tfrac{8}{3}x; 1-\eta) \quad (51) \end{aligned}$$

where $\zeta = (z_{13}z_{24}/z_{12}z_{23}z_{34}z_{14})$.

Actually, since we are interested in the limit $Q \to 1$, that is $x(Q) \to 0$, these simplify greatly to the order needed:[25]

$$\begin{aligned} Z_{aa} &= \left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}}\right)^{2x(Q)} (1+\tfrac{4\pi}{\sqrt{3}}x)\,{}_2F_1(-4x, \tfrac{1}{3}; \tfrac{2}{3}; 1-\eta) + O(x^2) \quad (52) \\ Z_{ab} &= \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})} (1-\eta)^{\frac{1}{3}}\,{}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; 1-\eta) + O(x) \quad (53) \end{aligned}$$

From (9), evaluated at $Q = 1$, now follows the main result for the crossing probability in the half-plane

$$P((z_1, z_2); (z_3, z_4)) = 1 - \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})} (1-\eta)^{\frac{1}{3}}\,{}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; 1-\eta) \quad (54)$$

which may also be written

$$P((z_1, z_2); (z_3, z_4)) = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})} \eta^{\frac{1}{3}}\,{}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; \eta) \quad (55)$$

Although this follows from standard identities on hypergeometric functions, there is a more physical argument, based on duality in bond percolation: whenever there is a cluster of open bonds connecting (z_1, z_2) to (z_3, z_4), there cannot be a cluster of open dual bonds connecting (z_2, z_3) to (z_4, z_1), and vice

[25]Notice that the ratio of gamma functions in Z_{aa} simplifies to $\sin\pi(\frac{2}{3}-\frac{8}{3}x)/\sin\pi(\frac{2}{3}+\frac{4}{3}x)$, so is easy to expand in powers of x.

versa. Therefore there is either a cluster of open bonds connecting (z_1, z_2) to (z_3, z_4), or a cluster of open dual bonds connecting (z_2, z_3) to (z_4, z_1). Thus

$$P(z_1, z_2; z_3, z_4) = 1 - P(z_2, z_3; z_4, z_1), \tag{56}$$

or, in terms of the cross-ratio,

$$P(\eta) = 1 - P(1 - \eta). \tag{57}$$

The expression for the mean number of crossing clusters is slightly more difficult, because we need to differentiate with respect to Q, that is, with respect to x. First we note that Z_{af} and Z_{fa} are equal to z_{12}^{-2x} and z_{34}^{-2x} respectively. Thus the ratio of partition functions in (12) may written entirely in terms of η

$$\frac{Z_{aa}Z_{ff}}{Z_{af}Z_{fa}} = (1-\eta)^{-2x(Q)}(1 + \tfrac{4\pi}{\sqrt{3}}x(Q))_2F_1(-4x(Q), \tfrac{1}{3}; \tfrac{2}{3}; 1-\eta) \tag{58}$$

to the order required. Thus

$$E[N_c] = x'(1)\left[-2\ln(1-\eta) + \tfrac{4\pi}{\sqrt{3}} - 4\sum_{n=1}^{\infty}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3}+n)}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}+n)}\frac{(1-\eta)^n}{n}\right] \tag{59}$$

where the series comes from differentiating ${}_2F_1(a, \ldots)$ with respect to its first argument a at $a = 0$.

Of course this result depends on the (presently) unknown value of $x'(1) \equiv (\mathrm{d}x/\mathrm{d}Q)|_{Q=1}$. This may be found[26] by noting that, as $\eta \to 0$, it is extremely unlikely that there is more than one crossing cluster, so that $E[N_c] \sim P \sim (\Gamma(\frac{2}{3})/\Gamma(\frac{4}{3})\Gamma(\frac{1}{3}))\eta^{\frac{1}{3}}$. Applying Stirling's formula to the coefficients of the series in (59) we see that they behave at large n like $(\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3}))n^{-\frac{4}{3}}$, which implies that the singular part of the sum of the series behaves like

[26]In fact, the complete dependence of $x(Q)$ has been conjectured using Coulomb gas methods.[3]

$(\Gamma(\frac{2}{3})\Gamma(-\frac{1}{3})/\Gamma(\frac{1}{3}))\eta^{\frac{1}{3}}$ as $\eta \to 0$. Equating this to the leading term in P then gives

$$x'(1) = -\left(4\Gamma(\tfrac{4}{3})\Gamma(-\tfrac{1}{3})\right)^{-1} = -\sin(4\pi/3)/(4\pi) = \sqrt{3}/(8\pi) \qquad (60)$$

This result for $x'(1)$ has a direct physical meaning. Consider once again the annular geometry, formed by sewing together the ends of a strip of width L and length W. The mean number of clusters crossing from one edge to the other is once again given by a formula like (12), where now the partition functions may be represented in the trace form like (36). In the limit $W \gg L$ the result is particularly simple, since $Z_{aa} \sim Z_{ff} \to 1$, while $Z_{af} = Z_{fa} \sim \exp(-\pi x(Q)(W/L))$. Thus in this limit,

$$E[N_c] \sim 2\pi x'(1)(W/L) = (\sqrt{3}/4)(W/L) \qquad (61)$$

It is to be expected that, for large W, $E[N_c]$ should be proportional to W, and therefore by scale invariance should go like (W/L), but the (irrational) universal coefficient is quite surprising.

7 Other approaches.

7.1 Crossing probability in the rectangle and modular invariance.

An important special case is when the curve Γ is a rectangle, with the segments γ_1 and γ_2 being opposite edges (see Fig. 6). The conformal mapping from the half-plane into the interior of the rectangle has the form of a Schwartz-Christoffel transformation

$$z \to w = \int_0^z \frac{dt}{(1-t^2)^{1/2}(1-k^2t^2)^{1/2}} \qquad (62)$$

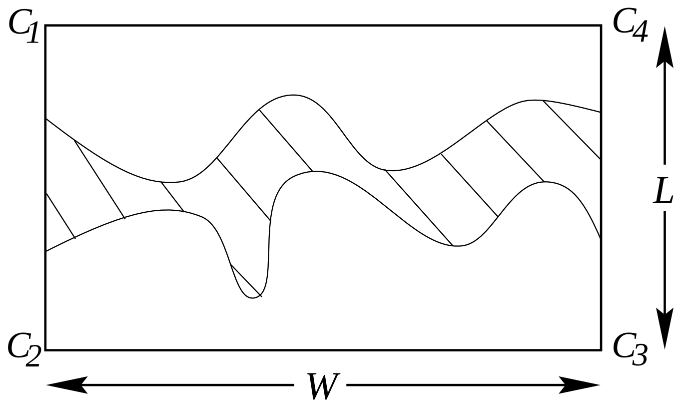

Figure 6: Crossing cluster in the rectangle.

which maps the points $(-k^{-1}, -1, 1, k^{-1})$ to the corners of the rectangle. The width and height of the rectangle are then

$$W = 2\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = 2K(k^2) \tag{63}$$

$$L = \int_1^{1/k} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}} = K(1-k^2) \tag{64}$$

where K is the complete elliptic integral of the first kind. The *aspect ratio* $r \equiv W/L = 2K(k^2)/K(1-k^2)$. Clearly it is possible numerically to solve this for k for a given r, and compute $\eta = ((1-k)/(1+k))^2$ and thence the crossing probability $P(r)$ via (2). However, it turns out that there is a more elegant formula, in terms of modular forms, due to Kleban[15] (see also Ziff[16].) Kleban and Zagier[17] have recently derived this result, using duality and some basic properties of conformal field theory, with one rather simple assumption, and we now summarise their argument.

From (9), $P(r) = 1 - Z_{ab}$, where Z_{ab} is the partition function for a strip of width L, with free boundary conditions on each edge, and length W. Think of evaluating this using the continuum version of the transfer matrix acting horizontally, along the strip:

$$Z_{ab} = \langle a|e^{-W\hat{H}_{ff}(L)}|b\rangle \tag{65}$$

where $\langle a|$ and $|b\rangle$ are so-called *boundary states*, whose form we shall not need. Inserting a complete set of eigenstates into this matrix element:

$$Z_{ab} = \sum_j \langle a|j\rangle\, q^{x_j/2} \langle j|b\rangle \tag{66}$$

where $q = e^{-2\pi(W/L)} = e^{-2\pi r}$ and the sum is over a complete set of scaling operators, consistent with the boundary conditions (ff) on the edges, the lowest of which has $x = 0$. The sum may be organised into a sum over primary operators, and their descendents, so that $P(r)$ has the form

$$P(r) = 1 - \sum_{x \in \pi_{ff}} q^{x/2} Q_x(q) \tag{67}$$

where π_{ff} is the set of primary scaling dimensions consistent with the boundary conditions (ff), and $Q_x(q)$ is a power series, starting with a term $O(q^0)$, representing the contribution of all the descendents. Note that in this case no odd powers of $q^{1/2}$ enter. This is a consequence of the reflection symmetry of the rectangle in the horizontal axis. Odd powers correspond to states which are odd under this operation, so do not couple to the states $|a\rangle$ and $|b\rangle$, which are even. On the other hand, we could also imagine computing Z_{ab} using the transfer matrix acting vertically. The result would have the form

$$Z_{ab} = \sum_{x \in \pi_{ab}} \tilde{q}^{x/2} R_x(\tilde{q}^{1/2}) \tag{68}$$

where $\tilde{q} \equiv e^{-2\pi/r}$ and now odd powers of $\tilde{q}^{1/2}$ are expected to enter because Z_{ab} is not symmetric under reflections in the vertical axis. Note that the lowest value of x in π_{ab} is the scaling dimension of $\phi_{a|b}$, which is strictly positive.

Finally, the function $P(r)$ also satisfies (57), which means that $P(r) = 1 - P(1/r)$. Comparing with (68) we see that $P(1/r)$ has the form of the series on the right hand side, and comparing this with (67) we see that in fact we may write

$$P(r) = \sum_{x \in \pi_{ab}} q^{x/2} Q_x(q) \tag{69}$$

Now assume that there is only one term in this sum, corresponding to $x = x_{a|b}$. Let $r = i\tau$, so that $q = e^{2\pi i \tau}$. Then P is holomorphic in the upper-half τ-plane. Define $f(\tau) = dP/d\tau$. The modular group is generated by the elements $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$, satisfying $(ST)^3 = 1$. Now (57) implies that under S, $f(-1/\tau) = -\tau^2 f(\tau)$, while under T, $f(\tau + 1) = e^{i\pi x} f(\tau)$. Since $(ST)^3 = 1$, $(e^{i\pi x})^3 = -1$, so that $x = m/3$ for some odd integer m. It follows that the function $g \equiv f^6$ satisfies $g(-1/\tau) = \tau^{12} g(\tau)$, $g(\tau+1) = g(\tau)$, and that it has a series expansion in positive integers powers of q. It is called a cusp form of weight 12. It turns out that there is only one such function, up to an overall multiplicative constant, which is $\eta(q)^{24}$, where η is the Dedekind function

$$\eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{70}$$

Thus

$$P(r) = \frac{2^{7/3}\pi^2}{\sqrt{3}\Gamma(\frac{1}{3})^3} \int_r^{\infty} \eta(ir')^4 dr' \tag{71}$$

where the constant is fixed by the requirement that $P(0) = 1$, and various identities satisfied by $\eta(q)$. Remarkably, this formula is identical[15] to the Schwartz-Christoffel transform of (2).

7.2 Carleson's formula for an equilateral triangle, and Smirnov's proof.

L. Carleson made the interesting observation that the crossing formula in an equilateral triangle has a much simpler form than that for a rectangle. This

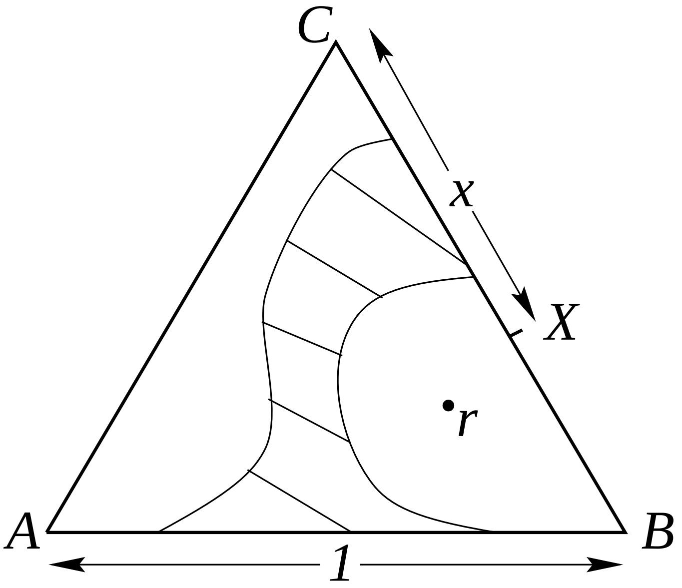

Figure 7: Crossing cluster in the equilateral triangle.

is because of a simple property of the hypergeometric differential equation arising from (40): because there is a solution with F = const., it follows that $F'(\eta)$ satisfies a simple first order equation which may be solved by quadrature. The result is that $P(\eta)$ may be written as

$$P(\eta) = \frac{\Gamma(\frac{2}{3})}{3\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})} \int_0^\eta (t(1-t))^{-2/3} dt \tag{72}$$

But the occurrence of the exponent $\frac{2}{3}$ means that this integral has just the form of the Schwartz-Christoffel conformal mapping from the upper half-plane to an equilateral triangle!

Thus if we consider an equilateral triangle ABC (see Fig. 7), whose side has a unit length, and consider the crossing probability from anywhere on side AB to the segment XC of length x of the side BC, then Carleson's version of the crossing formula is simply

$$P(AB, XC) = x \tag{73}$$

Very recently, S. Smirnov[6] has provided a *proof* of this formula starting from site percolation on the triangular lattice. He considers the probability $h(r)$ that a given point in the interior of the triangle is separated from the segment AC by at least one cluster spanning from AB to BC. Taking $r = X$ on the boundary segment BC gives the required crossing probability. He shows that $h(r)$ is, on the lattice, an approximate harmonic function, which, in the limit when the lattice spacing $a \to 0$, satisfies Laplace's equation with the boundary conditions that $h = 0$ on AC, $h = 1$ when $r = B$ and the components of ∇h at 60° to the boundary (i.e. parallel to AC) vanish along AB and BC. For the equilateral triangle, the solution of this is simple: $h(r)$ is the distance of the point r from the edge AC, in units where the height of the triangle is 1. Of course, in general the solution of this boundary value problem is conformally invariant, so in the half-plane, one gets the result (72). Smirnov's arguments explain why the crossing formula (2) is the boundary value of an analytic function, but do not immediately give insight into the conformal field theory structure underlying this result. It seems to be special to $Q = 1$.

7.3 The approach of Lawler, Schramm and Werner.

Finally we discuss briefly another approach initiated by Schramm[18], which fits into a larger investigation into the intersection properties of Brownian walks by the above authors (see Ref. [19] and many references therein) The idea is to study the properties of a percolation hull, or cluster boundary, which grows from the real axis up into the upper half plane. Suppose that all the edges along the negative real axis are open, and those on the positive real axis are closed. There is an infinitely large cluster formed by the union of the links on the negative real axis and those clusters attached to it. Consider exploring the boundary of this cluster with a time-dependent process which makes one step along this boundary in unit time. (The process is

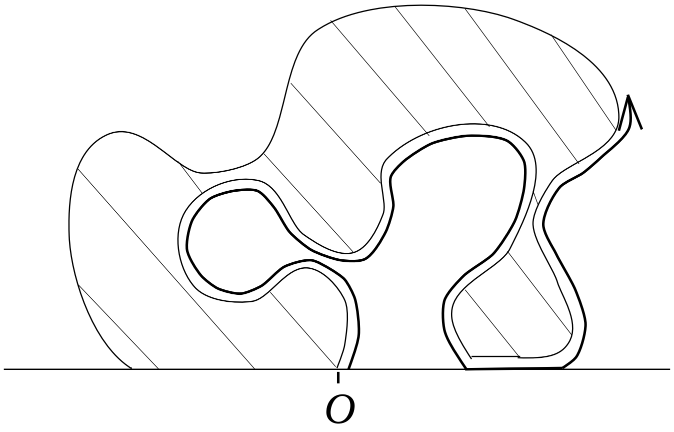

Figure 8: The path which explores the hull of a cluster attached to the negative real axis.

almost symmetric under $x \to -x$ because this path is also the boundary of a cluster of open dual links attached to the positive real axis.) The end of this path executes a non-crossing random walk in the upper half-plane, occasionally touching real axis (see Fig. 8). This successively excludes a larger and larger region of the half-plane. Schramm argued that the continuum limit of this process may be defined in terms of the conformal mapping $g_t(z)$ which maps the complement of the excluded region onto the upper half-plane. He conjectured[27] that this function satisfies a kind of stochastic ordinary differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - a(t)} \tag{74}$$

where $g_0(z) = z$ and $a(t)$ is a one-dimensional Brownian motion, i.e. $\dot{a} = \zeta(t)$ where ζ is a gaussian noise satisfying $\overline{\zeta(t)\zeta(t')} = \kappa\delta(t - t')$. This equation is defined up to the time that $g_t(z)$ hits $a(t)$: we can think of this as the time T_z at which the point z enters the excluded region.[28]

[27]The recent work of Smirnov[6] shows that percolation does indeed have a conformally invariant continuum limit. Uniqueness arguments then justify this conjecture *a posteriori*.

[28]It is useful to think about the case when $a(t) = 0$. The solution is then $g_t(z) = (z^2 + 4t)^{1/2}$. The end of the path lies at $2i\sqrt{t}$ at time t. The initial effect of the noise in

The continuum problem is conformally invariant by construction. If we now consider the crossing probability between the intervals $(-\infty,-a)$ and $(0,b)$, it may be related to the times T_{-a} and T_b by

$$P((-\infty,-a),(0,b)) = Pr(T_{-a} < T_b) \tag{75}$$

From the conjectured equation (74), with $\kappa = 6$,[29] it may be shown that the above is given by the crossing formula (2) with the cross-ratio $\eta = b/(a+b)$. Of course, many other related formulae also follow from this approach.

8 Conclusions.

The crossing formula (2) was first conjectured in 1992. The original argument was based on (1) the mapping to the Q-state Potts model in the limit $Q \to 1$ and (2) ideas of conformal field theory which had been developed up to that time, which have been the main subject of these lectures. However, it is clear that percolation is somewhat special as in some sense the measure, suitably defined, is conformally *invariant*, rather than merely covariant as is believed to occur in most critical systems. This is linked to the fact that for percolation the central charge c vanishes.[30] This invariance has allowed mathematicians to formulate other more direct approaches to the problem. It remains to be seen whether these methods will be successful in computing other quantities, such as the mean number of crossing clusters (3), since, at least in our approach, this requires going away from $Q=1$, and whether, in a more general context, they will shed light on the origin of conformal covariance in more general critical systems.

$a(t)$ is to give this an additional horizontal motion.

[29]Other values may be considered, which give generalised crossing formulas. In general, the process generated by (74) is called Stochastic Loewner Evolution with parameter κ (SLEκ).

[30]In general the partition function of a finite system in a region size L with smooth metric and smooth boundaries behaves like $Z \sim e^{c\chi L/6}$ where χ is the Euler character.[12] Thus the measure is not strictly even scale invariant when $c \neq 0$.

Acknowledgments. I would particularly like to thank Professors Y. Higuchi and M. Katori for inviting me to Japan to give these lectures, and for their kind hospitality. I have benefited from useful correspondence and conversations with P. Kleban, G. Lawler, O. Schramm, S. Smirnov and W. Werner on this and related subjects, and thank them for sending preprints of their work prior to publication. This work was supported in part by the EPSRC through grant GR/J 78327.

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