Introduction to Conformal Field Theory Trinity Term 2014 Prof. J. Cardy

Notes on the Conformal Bootstrap

These notes are an addendum to my 2008 les Houches lecture notes. For further details, see, eg,
`http://arxiv.org/pdf/1111.2115.pdf`
Global conformal invariance for any d constrains the 4-point function of the same (scalar) field to have the form

$$C = \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = x_{12}^{-2\Delta_\phi} x_{34}^{-2\Delta_\phi}\, g(u,v)$$

where g is a function of the (squared) cross-ratios $u = x_{12}^2 x_{34}^2/x_{13}^2 x_{24}^2$ and $v = x_{14}^2 x_{23}^2/x_{13}^2 x_{24}^2$. This 4-point function may be evaluated using the OPEs

$$\begin{aligned}
\phi(x_1)\phi(x_2) &= x_{12}^{-2\Delta_\phi} \sum_{\mathcal{O}} b_{\mathcal{O}} C_{\mathcal{O}}(x_{12}, \partial_{x_1})\mathcal{O}(x_1) \\
\phi(x_3)\phi(x_4) &= x_{34}^{-2\Delta_\phi} \sum_{\mathcal{O}'} b_{\mathcal{O}'} C_{\mathcal{O}}(x_{34}, \partial_{x_3})\mathcal{O}(x_3)
\end{aligned}$$

together with the 2-point functions $\langle\mathcal{O}(x_1)\mathcal{O}(x_3)\rangle$. In the above, the sums are over primary operators $\mathcal{O}$, which for $d > 2$ means operators corresponding to highest weight states of the global conformal group. All other operators are derviatives of these. It turns out that the functions $C_{\mathcal{O}}$ are entirely determined by this global conformal group, acting on both sides with the raising operators. Thus we can write

$$g(u,v) = \sum_{\mathcal{O}} b_{\mathcal{O}}^2 g_\Delta^\ell(u,v)$$

where the functions $g_\Delta^\ell(u,v)$ depend only on the dimensions and spins of the fields $\mathcal{O}$. In this context they are called conformal blocks. They are entirely determined by conformal invariance. The main point is that, by exchanging e.g. $x_1 \leftrightarrow x_3$, we must have, by crossing symmetry,

$$v^{\Delta_\phi} g(u,v) = u^{\Delta_\phi} g(v,u)$$

This looks like a tautology, but, like modular invariance, when combined with the expansion above, it is non-trivial. The reason it is called a 'bootstrap', is that we may imagine implementing it first for the fields with smallest dimension Δ_ϕ, and getting constraints on which Δs may occur as intermediate states. Then we can implement it for the four-point functions of these fields, and so on. The hope is that there will be only a denumerable number of solutions to the whole entire bootstrap. For $d = 2$ and $c < 1$, for example, we know this to be the case for other reasons.
Let use first see how it works there. In 2d it is better to think of the cross-ratio as being complex: $\eta = z_{12}z_{34}/z_{13}z_{24}$, etc. Then we can think of $g(u,v)$ as a function of η and $\bar\eta$.

Suppose for simplicity that ϕ is a Kac $\phi_{2,1}$ field. Then by the same arguments as in the SLE chapter, the 4-point function will satisfy a linear 2nd order pde, which translates into an ode as a function of η. It has regular singular points as the $x_{jk} \to 0$, that is at $\eta = 0, 1, \infty$, and is there a linear combination of 2 hypergometric functions.

$$g = \sum_{j,k=1,2} A_{jk} \mathcal{F}_j(\eta) \mathcal{F}_k(\bar{\eta})$$

where $\mathcal{F}_1(\eta) = 1 + O(\eta)$, $\mathcal{F}_2(\eta) \sim \eta^{\Delta_{3,1}}(1 + O(\eta))$. These two terms reflect the OPE $\phi_{1,2} \cdot \phi_{1,2} = 1 + \phi_{1,3}$. In the case of the Ising CFT $\mathcal{F}_{1,2} = (1 \pm (1-\eta)^{1/2})^{1/2}$. We can determine the coefficients in a number of ways. Since g should be single-valued under $\eta \to \eta e^{2\pi i}$, the off-diagonal terms should vanish. We can also do the same about $\eta = 1$. Alternatively, more in the spirit of the bootstrap, we can demend that it behave properly under $\eta \to 1 - \eta$. The result is that the only solution is

$$g = |\mathcal{F}_1(\eta)|^2 + |\mathcal{F}_2(\eta)|^2$$

A similar method works for all the mininmal models. Once we know the 4-point function we can extract the OPE coefficients. In the above case we find that $c_{12;12;13} = \frac{1}{2}$.
But in higher dimensions we can only hope either to solve it approximately, or to get bounds on the possible solutions.
It is useful to write the above relation as

$$u^{\Delta_\phi} - v^{\Delta_\phi} + \sum{}' b_{\mathcal{O}}^2 \left(u^{\Delta_\phi} g_\Delta^\ell(v,u) - v^{\Delta_\phi} g_\Delta^\ell(u,v) \right) = 0$$

where the sum does not include the contribution of the identity operator.
Since this true for any (u,v) we may expand around the symmetric point $u = v$, to any desired order. (This corresponds to the case when the 4 points lie at the vertices of a square.) This gives a system of linear equations with non-negative coefficients $b_{\mathcal{O}}^2$. The problem is reduced to a linear programming problem. Note that we can always truncate the expansion in Δ, equivalent to exploring the finite subspace where all the $b_{\mathcal{O}}$ for larger Δ vanish. If we then find no solution to these linear constraints for particular values of Δ_ϕ and Δs then we know there is no CFT with these values.
In 3D for scalar exchange $\ell = 0$ the conformal blocks are

$$g_\Delta(u,v) = u^{\Delta/2} \sum_{m,n=0}^{\infty} \frac{((\Delta/2)_m (\Delta/2)_{m+n})^2}{m!n!(\Delta - 1/2)_m (\Delta)_{2m+n}} u^m (1-v)^n$$

It is easy to see that, for a given Δ_ϕ, we cannot make the intermediate Δ's too big. As $\Delta \to \infty$, the conformal block $g(u,v) \sim u^{\Delta/2} e^{\Delta u} e^{\Delta(1-v)}$.
So we should have

$$v^{\Delta_\phi}(1 + \sum b_{\mathcal{O}}^2 u^{\Delta/2} e^{\Delta u} e^{\Delta(1-v)})$$

being symmetric under $v \leftrightarrow u$. Since this is a function which decays as a power times something growing exponentially, this cannot happen. So there should be a lower bound on Δ for a given Δ_ϕ. We also see from this that the bound will less stringent as Δ_ϕ increases.

The linear programming approach is completely rigorous but at some point we have to resort to numerics. The results are shown below, for 3d CFT which has the properties expected of the scaling limit of the 3d Ising model: a Z_2 odd primary σ, a Z_2 even primary ϵ, and an irrelevant primary ϵ', corresponding to the fields ϕ, ϕ^2 and ϕ^4 in a ϕ^4 field theory. With minimal assumptions about $\Delta_{\epsilon'}$ one finds the bounds shown below. Remarkably, the best known values for the 3d Ising model lie very close to the 'corner' on the boundary. This is presumably telling us something either exact or almost exact about the CFT.