Solutions 2

1. Use the transformation $w = (L/2\pi)\log z$ as for the 2-point function, and the fact that in the plane $\langle\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle = c_{ijk}/((z_1-z_2)^{\Delta_i+\Delta_j-\Delta_k}\cdots)$. On the cylinder, taking $w_2 = 0$, and $w_1 = u_1 + iv_1$, $w_3 = u_3 + iv_3$, with $u_1 \ll -L$, $u_3 \gg L$, we find after some algebra

$$\begin{aligned}&\langle\phi_i(u_1,v_1)\phi_j(0,0)\phi_k(u_3,v_3)\rangle_{\rm cyl}\\&\sim c_{ijk}(2\pi/L)^{x_i+x_j+x_k}e^{-2\pi x_i|u_1|/L}e^{2\pi i(s_i-s_j)v_1/L}e^{-2\pi x_k u_3/L}e^{2\pi i(s_j-s_k)v_3/L}\end{aligned}$$

where $x_i, s_i = \Delta_i \pm \bar\Delta_i$, etc. If we write this in terms of operators

$$\begin{aligned}&\langle 0|e^{iv_1\hat P}\hat\phi_i(0)e^{-iv_1\hat P}e^{-|u_1|\hat H}\hat\phi_j(0)e^{-u_3\hat H}e^{iv_3\hat P}\hat\phi_k(0)e^{-iv_3\hat P}|0\rangle\\&\sim\langle 0|\phi_i\rangle\langle\phi_i|\hat\phi_j|\phi_k\rangle\langle\phi_k|0\rangle e^{-2\pi x_i|u_1|/L}e^{2\pi i(s_i-s_j)v_1/L}e^{-2\pi x_k u_3/L}e^{2\pi i(s_j-s_k)v_3/L}\end{aligned}$$

On the other hand doing the same for the 2-point function (or simply setting $\phi_j = 1$ in the above) we see that $\langle 0|\phi_i\rangle = (2\pi/L)^{x_i}$, $\langle\phi_k|0\rangle = (2\pi/L)^{x_k}$, so finally

$$\langle\phi_i|\hat\phi_j(0)|\phi_k\rangle = (2\pi/L)^{x_j}\,c_{ijk}$$

This means that the OPE coefficients may be measured knowing matrix elements on the cylinder.

2. According to first-order perturbation theory, we need to work out

$$\begin{aligned}&\lambda\langle\phi_j|\int_0^L T(v)\overline{T}(v)dv|\phi_j\rangle = \lambda\int\langle\phi_j|T(v)|\phi_j\rangle\langle\phi_j|\overline{T}(v)|\phi_j\rangle dv\\&=\lambda(1/L)\langle\phi_j|\int T(v)dv|\phi_j\rangle\langle\phi_j|\int\overline{T}(v)dv|\phi_j\rangle\\&=\lambda(2\pi/L)^2(1/L)\langle\phi_j|L_0-(c/24)|\phi_j\rangle\langle\phi_j|\overline{L}_0-(c/24)|\phi_j\rangle\\&=\lambda(4\pi^2/L^3)(\Delta_j-(c/24))(\bar\Delta_j-(c/24))\end{aligned}$$

Note this is negligible compared to the $O(1/L)$ term as $L\to\infty$, typical of the effect of an irrelevant perturbation.

3. Suppose the operator is ϕ. Its fusion rules must have the form $\phi\cdot\phi = 1+\phi$, with no other operators. Since this is a minimal model, it must lie in the first column or first row of the Kac table. In the first case it must be $\phi_{1,2}$, but in general this will also give $\phi_{1,3}$ in fusion with itself. This can only work therefore if in fact $\Delta_{1,3} = \Delta_{1,2}$, that is $(p-2p')^2 = (p-3p')^2$, that is $2p - 5p'$, or $p = 5$, $p' = 2$. (No multiples are allowed because $p' > 2$ would allow more columns in the Kac table.) Thus $c = -\frac{22}{5}$ and $\Delta_{1,2} = -\frac{1}{5}$. We also find that $\Delta_{1,4} = 1$. This was identified in [JC, Phys. Rev.

Lett. 54, 1354, (1985)] as the scaling limit of the Yang-Lee edge singularity (a ϕ^3 field theory with purely imaginary coupling.)

Exchanging rows and columns just swaps $p \leftrightarrow p'$. The case of two fields is slightly more complex: if we allow only one column then we get the model with $p = 7, p' = 2$. If we allow two columns we get the unitary model with $c = \frac{1}{2}$.

4. Modular invariance implies that $Z(\delta) = Z(1/\delta)$, and hence that $Z'(1) = 0$. Writing $Z = e^{-2\pi c\delta/12} \sum_j e^{-2\pi x_j \delta}$ gives

$$\frac{c}{12} = \frac{\sum_j x_j e^{-2\pi x_j}}{\sum_j e^{-2\pi x_j}} < x_{\min} \frac{\sum_{j\geq 1} e^{-2\pi x_j}}{1 + \sum_{j\geq 1} e^{-2\pi x_j}} < x_{\min}$$

but it is clearly possible to do much better than this. For the latest effort in this direction see arXiv:1405.5137

5. The answer is almost given: $\mathrm{Tr}\,\Sigma\, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} = |\chi_0|^2 + |\chi_{1/2}|^2 - |\chi_{1/16}|^2$. Writing this as the modular invariant sum minus $2|\chi_{1/16}|^2$, we see that under S it goes into

$$\begin{aligned} &|\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2 - 2\tfrac{1}{\sqrt{2}}(\overline{\chi_0} - \overline{\chi_{1/2}})\tfrac{1}{\sqrt{2}}(\chi_0 - \chi_{1/2}) \\ &= \overline{\chi}_0 \chi_{1/2} + \overline{\chi}_{1/2} \chi_0 + |\chi_{1/16}|^2 \end{aligned}$$

This means that in the antiperiodic sector the lowest energy state corresponds to an operators with dimensions $(\frac{1}{16}, \frac{1}{16})$. This is the disorder operator, dual to the magnetisation and with the same dimensions. There are also a primaries with dimensions $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ — these are the Ising fermions.