Advanced Topics in Statistical Mechanics
Michaelmas Term, 2007 – Prof. J. Cardy
Homework Problem Solutions

1. Let $f_0 = (e^{(\epsilon-\mu)/T}+1)^{-1}$ be the unperturbed Fermi distribution. The Boltzmann equation to first order in $f-f_0$ is then

$$v\frac{\partial f_0}{\partial x} = -\frac{f-f_0}{\tau}$$

The LHS is

$$vf_0'(\epsilon)\left[-\frac{\epsilon}{T}\frac{\partial T}{\partial x} - T\frac{\partial(\mu/T)}{\partial x}\right]$$

The particle current is

$$J_x = \int v_x f\frac{2d^3p}{h^3}$$

and since this comes entirely from the difference $f-f_0$ we can use the BE to write it as

$$J_x = -\tau\int v_x^2 f_0'(\epsilon)\Big[\cdots\Big]D(\epsilon)d\epsilon$$

where the expression in square brackets is the same as above, and D is the density of states.

On the other hand, the energy current Q_x is given by the same expression with the insertion of a factor of ϵ (or $\epsilon-\epsilon_F$ – it depends if you prefer to think about electrons and holes) in the integrand. We should adjust $\partial(\mu/T)/\partial x$ so that $J_x=0$. [The heat current is actually $Q_x-\mu J_x$, but since $J_x=0$ they are the same.] In the approximation that $f_0'(\epsilon)\propto\delta(\epsilon-\epsilon_F)$, this insertion is a constant ϵ_F, and so $Q_x=0$ also. In order to get a non-zero answer, we must do better.

If we assume a spherical Fermi surface with $\epsilon=\mathbf{p}^2/2m$, then $D(\epsilon)=(2\epsilon m^3)^{1/2}/\pi^2h^3$ and $\overline{v_x^2}=2\epsilon/3m$. Then (I apologise for not using the notation of Ashcroft and Mermin – my copy is missing so I had to work it out by myself)

$$Q_x = \frac{2^{3/2}m^{1/2}}{3\pi^2h^3}\left(-I_{7/2}\frac{1}{T}\frac{\partial T}{\partial x} - I_{5/2}\frac{\partial(\mu/T)}{\partial x}\right)$$

while

$$J_x = \frac{2^{3/2}m^{1/2}}{3\pi^2 h^3}\left(-I_{5/2}\frac{1}{T}\frac{\partial T}{\partial x} - I_{3/2}\frac{\partial(\mu/T)}{\partial x}\right)$$

where

$$I_s = -\int_0^\infty \epsilon^s f_0'(\epsilon)d\epsilon$$

Imposing $J_x = 0$ we get

$$Q_x = -\frac{2^{3/2}m^{1/2}}{3\pi^2 h^3}\left(I_{7/2} - \frac{I_{5/2}^2}{I_{3/2}}\right)\frac{1}{T}\frac{\partial T}{\partial x}$$

To get a non-zero result at low T we must write $\epsilon = \epsilon_F + \delta\epsilon$ and expand to second order in $\delta\epsilon$. We therefore get $I_s = n(1 + \text{const}.s(s-1)T^2)$. The final answer (after some algebra) is $Q_x = \lambda(-\partial T/\partial x)$ where

$$\lambda = \frac{\pi^2 n\tau}{3m}T\,.$$

2. Let us first give the heuristic derivation. The effective field acting on S_i is $h_i^{eff} = \sum_j J_{ij}m_j + h$, where $m_j = \tanh(h_j^{eff})$. We now assume that the h_i^{eff} are gaussian random variables with mean h and variance (as before) $J^2 q$. Self-consistency now demands that

$$q \propto \int dh^{eff} e^{-(h^{eff}-h)^2/2J^2 q}\tanh^2(\beta h^{eff})$$

On rescaling $h^{eff} - h = J\sqrt{q}z$,

$$q = (2\pi)^{-1/2}\int dz e^{-z^2/2}\tanh^2\beta(J\sqrt{q}z + h)$$

Similarly, the mean magnetisation is

$$M = (2\pi)^{-1/2}\int dz e^{-\frac{1}{2}z^2}\tanh\beta(J\sqrt{q}z + h)$$

In the replica method, the modification is also straightforward: it comes when we make the trace over the S_i^α at a single site:

$$\text{Tr}(2\pi)^{-1/2}\int dz e^{-\frac{1}{2}z^2 + (z(\beta J)\sqrt{q}+h)\sum_\alpha S_i^\alpha}$$

[I apologise that the last part wasn't totally clear: the last words should read 'diverges as $T \to T_c+$'.] To find the dependence of M on h, we first have to solve for q as a function of h. The SK equation for small q and small h takes the form

$$q = (\beta J)q - q^2 + O(h^2)$$

In the high temperature phase we can ignore the q^2 term so

$$q \sim \frac{h^2}{T - T_c}$$

If we now expand the equation for M we find

$$M \sim \int dz e^{-\frac{1}{2}z^2} \left(\beta(J\sqrt{q}z + h) + \text{const}(J\sqrt{q}z + h)^3 + \cdots\right)$$

The non-zero terms are of the form (apart from constants) $h + qh + h^3$. So we see that $\partial M/\partial h$ is finite as $t \to T_c$, but

$$\frac{\partial^3 M}{\partial h^3} \propto \frac{1}{T - T_c}$$

3. (a) The 2-point function is

$$\langle \cos p(\theta(r_1) - \phi) \cos p(\theta(r_2) - \phi) \rangle \propto \text{Re}\, \langle e^{ip(\theta(r_1) - \theta(r_2))} \rangle$$

We did the case $p = 1$ in the lecture. In general we get

$$e^{-\frac{1}{2}p^2 \langle (\theta(r_1) - \theta(r_2))^2 \rangle} \sim \frac{1}{|r_1 - r_2|^{p^2/2\pi K}}$$

This means that $x_p = p^2/4\pi K$. If we add this term to the hamiltonian, as

$$h_p \int \frac{d^2 r}{a^2} \cos p(\theta(r) - \phi)$$

and make an RG transformation $a \to ba$, we see that $h_p \to b^{y_p} h_p$ where

$$y_p = 2 - x_p = 2 - \frac{p^2}{4\pi K}$$

(b) This is irrelevant if $y_p < 0$, i.e. $K < p^2/\pi$, or $T > T_p = 8\pi J/p^2$. On the other hand the vortices are irrelevant if $T < T_{KT} = \pi J/2$. There is therefore a range of temperatures where both are irrelevant if $p > 4$. This will have quasi-LRO. If $T < T_p$, h_p is relevant and the system will order into one of p possible phases. (In this case we can expand the $\cos p\theta$ about one of the maxima and get a quadratic term $\propto \theta^2$ which corresponds to a finite correlation length.) If $T > T_{KT}$ we expect the usual paramagnetic phase. If $p \leq 4$ the system will undergo a single transition from the ordered phase to the paramagnetic phase, with no quasi-LRO intermediate phase.

(c) For $h_p \gg J$, the spins should follow the local random field. For $h \ll J$ we might expect an ordered phase, and for small T a quasi-LRO ordered phase, just as for $h_p = 0$.

(d) The replicated partition function has the form

$$\mathrm{Tr} \exp\left(- \int d^2r (\frac{1}{2} K \sum_\alpha (\nabla\theta_\alpha)^2 + h_p \sum_\alpha \cos p(\theta_\alpha - \phi(r)))\right)$$

Performing the quenched average by expanding in h_p, integrating over $\phi(r)$ and re-exponentiating, we get

$$\exp\left(\Delta_p \sum_{\alpha\neq\beta} \cos p(\theta_\alpha - \theta_\beta)\right)$$

If we work out the 2-point function of this, we just get the square of the result in part (1), so the scaling dimension is $2x_p = p^2/2\pi K$. (This is just as in the Harris criterion RG argument.) Hence the RG eigenvalue of Δ_p is

$$2 - \frac{p^2}{2\pi K}$$

and is now irrelevant for $T > \frac{1}{2}T_p = 4\pi J/p^2$. Both this and the vortices are irrelevant for $p > 2\sqrt{2}$.

[Actually the analysis is much more complicated than this because other terms get generated in the RG, like $\sum_{\alpha\neq\beta}(\nabla\theta_\alpha)(\nabla\theta_\beta)$. See J. Cardy and S. Ostlund, Phys. Rev. B **25**, 6899 (1981).]