Introduction to Quantum Mechanics HT 2010
Problems 7 (Easter vacation)

7.1* By expressing the annihilation operator A of the harmonic oscillator in the momentum representation, obtain $\langle p|0\rangle$. Check that your expression agrees with that obtained from the Fourier transform of

$$\langle x|0\rangle = \frac{1}{(2\pi\ell^2)^{1/4}}\,\mathrm{e}^{-x^2/4\ell^2}, \quad \text{where} \quad \ell \equiv \sqrt{\frac{\hbar}{2m\omega}}. \tag{7.1}$$

7.2 Show that for any two $N\times N$ matrices A, B, trace$([A,B]) = 0$. Comment on this result in the light of the results of Problem 3.7 and the canonical commutation relation $[x,p] = \mathrm{i}\hbar$.

7.3* A **Fermi oscillator** has Hamiltonian $H = f^\dagger f$, where f is an operator that satisfies

$$f^2 = 0 \qquad ; \qquad ff^\dagger + f^\dagger f = 1. \tag{7.2}$$

Show that $H^2 = H$, and thus find the eigenvalues of H. If the ket $|0\rangle$ satisfies $H|0\rangle = 0$ with $\langle 0|0\rangle = 1$, what are the kets (a) $|a\rangle \equiv f|0\rangle$, and (b) $|b\rangle \equiv f^\dagger|0\rangle$?

In quantum field theory the vacuum is pictured as an assembly of oscillators, one for each possible value of the momentum of each particle type. A boson is an excitation of a harmonic oscillator, while a fermion in an excitation of a Fermi oscillator. Explain the connection between the spectrum of $f^\dagger f$ and the Pauli principle.

7.4 In the time interval $(t+\delta t, t)$ the Hamiltonian H of some system varies in such a way that $|H|\psi\rangle|$ remains finite. Show that under these circumstances $|\psi\rangle$ is a continuous function of time.

A harmonic oscillator with frequency ω is in its ground state when the stiffness of the spring is instantaneously reduced by a factor $f^4 < 1$, so its natural frequency becomes $f^2\omega$. What is the probability that the oscillator is subsequently found to have energy $\frac{3}{2}\hbar f^2\omega$? Discuss the classical analogue of this problem.

7.5* P is the probability that at the end of the experiment described in Problem 7.4, the oscillator is in its second excited state. Show that when $f = \frac{1}{2}$, $P = 0.144$ as follows. First show that the annihilation operator of the original oscillator

$$A = \tfrac{1}{2}\left\{(f^{-1}+f)A' + (f^{-1}-f)A'^\dagger\right\}, \tag{7.3}$$

where A' and $A'^\dagger$ are the annihilation and creation operators of the final oscillator. Then writing the ground-state ket of the original oscillator as a sum $|0\rangle = \sum_n c_n|n'\rangle$ over the energy eigenkets of the final oscillator, impose the condition $A|0\rangle = 0$. Finally use the normalisation of $|0\rangle$ and the orthogonality of the $|n'\rangle$. What value do you get for the probability of the oscillator remaining in the ground state?

Show that at the end of the experiment the expectation value of the energy is $0.2656\hbar\omega$. Explain physically why this is less than the original ground-state energy $\frac{1}{2}\hbar\omega$.

This example contains the physics behind the inflationary origin of the Universe: gravity explosively enlarges the vacuum, which is an infinite collection of harmonic oscillators (Problem 7.3). Excitations of these oscillators correspond to elementary particles. Before inflation the vacuum is unexcited so every oscillator is in its ground state. At the end of inflation, there is non-negligible probability of many oscillators being excited and each excitation implies the existence of a newly created particle.

7.6* Let $B = cA + sA^\dagger$, where $c \equiv \cosh\theta$, $s \equiv \sinh\theta$ with θ a real constant and A, $A^\dagger$ are the usual ladder operators. Show that $[B, B^\dagger] = 1$.

Consider the Hamiltonian

$$H = \epsilon A^\dagger A + \tfrac{1}{2}\lambda(A^\dagger A^\dagger + AA), \tag{7.4}$$

where ϵ and λ are real and such that $\epsilon > \lambda > 0$. Show that when

$$\epsilon c - \lambda s = Ec \quad ; \quad \lambda c - \epsilon s = Es \tag{7.5}$$

with E a constant, $[B,H] = EB$. Hence determine the spectrum of H in terms of ϵ and λ.

7.7 Verify that $[\mathbf{J}, \mathbf{x}\cdot\mathbf{x}] = 0$ and $[\mathbf{J}, \mathbf{x}\cdot\mathbf{p}] = 0$ by using the commutation relations $[x_i, J_j] = \mathrm{i}\sum_k \epsilon_{ijk}x_k$ and $[p_i, J_j] = \mathrm{i}\sum_k \epsilon_{ijk}p_k$.

7.8* The matrix for rotating an ordinary vector by ϕ around the z axis is

$$\mathbf{R}(\phi) \equiv \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.6}$$

From $\mathbf{R}$ calculate the matrix $\boldsymbol{\mathcal{J}}_z$ that appears in $\mathbf{R}(\phi) = \exp(-\mathrm{i}\boldsymbol{\mathcal{J}}_z\phi)$. Introduce new coordinates $u_1 \equiv (x - \mathrm{i}y)/\sqrt{2}$, $u_2 = z$ and $u_3 \equiv (x + \mathrm{i}y)/\sqrt{2}$. Write down the matrix $\mathbf{M}$ that appears in $\mathbf{u} = \mathbf{M}\cdot\mathbf{x}$ [where $\mathbf{x} \equiv (x, y, z)$] and show that it is unitary. Then show that

$$\boldsymbol{\mathcal{J}}'_z \equiv \mathbf{M}\cdot\boldsymbol{\mathcal{J}}_z\cdot\mathbf{M}^\dagger. \tag{7.7}$$

is identical with S_z in the set of spin-one Pauli analogues

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{7.8}$$

Write down the matrix $\boldsymbol{\mathcal{J}}_x$ whose exponential generates rotations around the x axis, calculate $\boldsymbol{\mathcal{J}}'_x$ by analogy with equation (7.7) and check that your result agrees with S_x in the set (7.8). Explain as fully as you can the meaning of these calculations.

7.9 Determine the commutator $[\boldsymbol{\mathcal{J}}'_x, \boldsymbol{\mathcal{J}}'_z]$ of the generators used in Problem 7.8. Show that it is equal to $-\mathrm{i}\boldsymbol{\mathcal{J}}'_y$, where $\boldsymbol{\mathcal{J}}'_y$ is identical with S_y in the set (7.8).

7.10* In this problem you derive the wavefunction

$$\langle\mathbf{x}|\mathbf{p}\rangle = \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}/\hbar} \tag{7.9}$$

of a state of well defined momentum from the properties of the translation operator $U(\mathbf{a})$. The state $|\mathbf{k}\rangle$ is one of well-defined momentum $\hbar\mathbf{k}$. How would you characterise the state $|\mathbf{k}'\rangle \equiv U(\mathbf{a})|\mathbf{k}\rangle$? Show that the wavefunctions of these states are related by $u_{\mathbf{k}'}(\mathbf{x}) = \mathrm{e}^{-\mathrm{i}\mathbf{a}\cdot\mathbf{k}}u_{\mathbf{k}}(\mathbf{x})$ and $u_{\mathbf{k}'}(\mathbf{x}) = u_{\mathbf{k}}(\mathbf{x} - \mathbf{a})$. Hence obtain equation (7.9).

7.11 An electron moves along an infinite chain of potential wells. For sufficiently low energies we can assume that the set $\{|n\rangle\}$ is complete, where $|n\rangle$ is the state of definitely being in the n^{th} well. By analogy with our analysis of the NH_3 molecule we assume that for all n the only non-vanishing matrix elements of the Hamiltonian are $\mathcal{E} \equiv \langle n|H|n\rangle$ and $A \equiv \langle n \pm 1|H|n\rangle$. Give physical interpretations of the numbers A and $\mathcal{E}$.

Explain why we can write

$$H = \sum_{n=-\infty}^{\infty} \mathcal{E}|n\rangle\langle n| + A\left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right). \tag{7.10}$$

Writing an energy eigenket $|E\rangle = \sum_n a_n|n\rangle$ show that

$$a_m(E - \mathcal{E}) - A\left(a_{m+1} + a_{m-1}\right) = 0. \tag{7.11}$$

Obtain solutions of these equations in which $a_m \propto \mathrm{e}^{\mathrm{i}km}$ and thus find the corresponding energies E_k. Why is there an upper limit on the values of k that need be considered?

Initially the electron is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|E_k\rangle + |E_{k+\Delta}\rangle\right), \tag{7.12}$$

where $0 < k \ll 1$ and $0 < \Delta \ll k$. Describe the electron's subsequent motion in as much detail as you can.

7.12 ^{238}U decays by α emission with a mean lifetime of 6.4 Gyr. Take the nucleus to have a diameter $\sim 10^{-14}$ m and suppose that the α particle has been bouncing around within it at speed $\sim c/3$. Modelling the potential barrier that confines the α particle to be a square one of height V_0 and width $2a$, give an order-of-magnitude estimate of $W = (2mV_0a^2/\hbar^2)^{1/2}$. Given that the energy released by the decay is ~ 4 MeV and the atomic number of uranium is $Z = 92$, estimate the width of the barrier through which the α particle has to tunnel. Hence give a very rough estimate of the barrier's typical height. Outline numerical work that would lead to an improved estimate of the structure of the barrier.

7.13* Particles of mass m and momentum $\hbar k$ at $x < -a$ move in the potential

$$V(x) = V_0 \begin{cases} 0 & \text{for } x < -a \\ \frac{1}{2}[1 + \sin(\pi x/2a)] & \text{for } |x| < a \\ 1 & \text{for } x > a, \end{cases} \tag{7.13}$$

where $V_0 < \hbar^2k^2/2m$. Numerically reproduce the reflection probabilities plotted Figure 5.20 as follows. Let $\psi_i \equiv \psi(x_j)$ be the value of the wavefunction at $x_j = j\Delta$, where Δ is a small increment in the x coordinate. From the TISE show that

$$\psi_j \simeq (2 - \Delta^2k^2)\psi_{j+1} - \psi_{j+2}, \tag{7.14}$$

where $k \equiv \sqrt{2m(E - V)}/\hbar$. Determine ψ_j at the two grid points with the largest values of x from a suitable boundary condition, and use the recurrence relation (7.14) to determine ψ_j at all other grid points. By matching the values of ψ at the points with the smallest values of x to a sum of sinusoidal waves, determine the probabilities required for the figure. Be sure to check the accuracy of your code when $V_0 = 0$, and in the general case explicitly check that your results are consistent with equal fluxes of particles towards and away from the origin.

Equation (11.40) gives an analytical approximation for ψ in the case that there is negligible reflection. Compute this approximate form of ψ and compare it with your numerical results for larger values of a.

7.14* We have that

$$L_+ \equiv L_x + \mathrm{i}L_y = \mathrm{e}^{\mathrm{i}\phi}\Big(\frac{\partial}{\partial\theta} + \mathrm{i}\cot\theta\frac{\partial}{\partial\phi}\Big). \tag{7.15}$$

From the Hermitian nature of $L_z = -\mathrm{i}\partial/\partial\phi$ we infer that derivative operators are anti-Hermitian. So using the rule $(AB)^\dagger = B^\dagger A^\dagger$ on equation (7.15), we infer that

$$L_- \equiv L_+^\dagger = \Big(-\frac{\partial}{\partial\theta} + \mathrm{i}\frac{\partial}{\partial\phi}\cot\theta\Big)\mathrm{e}^{-\mathrm{i}\phi}.$$

This argument and the result it leads to is wrong. Obtain the correct result by integrating by parts $\int \mathrm{d}\theta\,\sin\theta \int \mathrm{d}\phi\,(f^*L_+g)$, where f and g are arbitrary functions of θ and ϕ. What is the fallacy in the given argument?

7.15* By writing $\hbar^2L^2 = (\mathbf{x}\times\mathbf{p})\cdot(\mathbf{x}\times\mathbf{p}) = \sum_{ijklm}\epsilon_{ijk}x_jp_k\,\epsilon_{ilm}x_lp_m$ show that

$$p^2 = \frac{\hbar^2L^2}{r^2} + \frac{1}{r^2}\left\{(\mathbf{r}\cdot\mathbf{p})^2 - \mathrm{i}\hbar\mathbf{r}\cdot\mathbf{p}\right\}. \tag{7.16}$$

By showing that $\mathbf{p}\cdot\hat{\mathbf{r}} - \hat{\mathbf{r}}\cdot\mathbf{p} = -2\mathrm{i}\hbar/r$, obtain $\mathbf{r}\cdot\mathbf{p} = rp_r + \mathrm{i}\hbar$. Hence obtain

$$p^2 = p_r^2 + \frac{\hbar^2L^2}{r^2}. \tag{7.17}$$

Give a physical interpretation of one over $2m$ times this equation.

7.16 A system that has total orbital angular momentum $\sqrt{6}\hbar$ is rotated through an angle ϕ around the z axis. Write down the 5×5 matrix that updates the amplitudes a_m that L_z will take the value m.

7.17 Write down the expression for the commutator $[\sigma_i, \sigma_j]$ of two Pauli matrices. Show that the anticommutator of two Pauli matrices is

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}. \tag{7.18}$$

7.18 Tritium, ^{3}H, is highly radioactive and decays with a half-life of 12.3 years to ^{3}He by the emission of an electron from its nucleus. The electron departs with 16 keV of kinetic energy. Explain why its departure can be treated as sudden in the sense that the electron of the original tritium atom barely moves while the ejected electron leaves.

Calculate the probability that the newly-formed ^{3}He atom is in an excited state. Hint: evaluate $\langle 1,0,0;Z=2|1,0,0;Z=1\rangle$.

7.19* A spherical potential well is defined by

$$V(r)=\begin{cases}0 & \text{for } r<a\\ V_0 & \text{otherwise,}\end{cases} \tag{7.19}$$

where $V_0>0$. Consider a stationary state with angular-momentum quantum number l. By writing the wavefunction $\psi(\mathbf{x})=R(r)\mathrm{Y}_l^m(\theta,\phi)$ and using $p^2=p_r^2+\hbar^2L^2/r^2$, show that the state's radial wavefunction $R(r)$ must satisfy

$$-\frac{\hbar^2}{2m}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r}\right)^2R+\frac{l(l+1)\hbar^2}{2mr^2}R+V(r)R=ER. \tag{7.20}$$

Show that in terms of $S(r)\equiv rR(r)$, this can be reduced to

$$\frac{\mathrm{d}^2S}{\mathrm{d}r^2}-l(l+1)\frac{S}{r^2}+\frac{2m}{\hbar^2}(E-V)S=0. \tag{7.21}$$

Assume that $V_0>E>0$. For the case $l=0$ write down solutions to this equation valid at (a) $r<a$ and (b) $r>a$. Ensure that R does not diverge at the origin. What conditions must S satisfy at $r=a$? Show that these conditions can be simultaneously satisfied if and only if a solution can be found to $k\cot ka=-K$, where $\hbar^2k^2=2mE$ and $\hbar^2K^2=2m(V_0-E)$. Show graphically that the equation can only be solved when $\sqrt{2mV_0}\,a/\hbar>\pi/2$. Compare this result with that obtained for the corresponding one-dimensional potential well.

The deuteron is a bound state of a proton and a neutron with zero angular momentum. Assume that the strong force that binds them produces a sharp potential step of height V_0 at interparticle distance $a=2\times10^{-15}$ m. Determine in MeV the minimum value of V_0 for the deuteron to exist. Hint: remember to consider the dynamics of the *reduced* particle.

7.20* Given that the ladder operators for hydrogen satisfy

$$A_l^\dagger A_l=\frac{a_0^2\mu}{\hbar^2}H_l+\frac{Z^2}{2(l+1)^2}\quad\text{and}\quad[A_l,A_l^\dagger]=\frac{a_0^2\mu}{\hbar^2}(H_{l+1}-H_l), \tag{7.22}$$

where H_l is the Hamiltonian for angular-momentum quantum number l, show that

$$A_{l-1}A_{l-1}^\dagger=\frac{a_0^2\mu}{\hbar^2}H_l+\frac{Z^2}{2l^2}. \tag{7.23}$$

Hence show that

$$A_{l-1}^\dagger|E,l\rangle=\frac{Z}{\sqrt{2}}\left(\frac{1}{l^2}-\frac{1}{n^2}\right)^{1/2}|E,l-1\rangle, \tag{7.24}$$

where n is the principal quantum number. Explain the physical meaning of this equation and its use in setting up the theory of the hydrogen atom.

7.21* From equation (8.42) show that $l'+\frac12=\sqrt{(l+\frac12)^2-\beta}$ and that the increment Δ in l' when l is increased by one satisfies $\Delta^2+\Delta(2l'+1)=2(l+1)$. By considering the amount by which the solution of this equation changes when l' changes from l as a result of β increasing from zero to a small number, show that

$$\Delta=1+\frac{2\beta}{4l^2-1}+\mathrm{O}(\beta^2). \tag{7.25}$$

Explain the physical significance of this result.

7.22 Show that Ehrenfest's theorem yields equation (8.66) with $\mathbf{B}=0$ as the classical equation of motion of the vector $\mathbf{S}$ that is implied by the spin–orbit Hamiltonian (8.67).