Introduction to Quantum Mechanics HT 2012
Problems 5 (week 3)

5.1 Show that $\langle j,j|J_x|j,j\rangle = \langle j,j|J_y|j,j\rangle = 0$ and that $\langle j,j|(J_x^2 + J_y^2)|j,j\rangle = j$. Discuss the implications of these results for the uncertainty in the orientation of the classical angular momentum vector $\mathbf{J}$ for both small and large values of j.

5.2 Write down the expression for the commutator $[\sigma_i, \sigma_j]$ of two Pauli matrices. Show that the anticommutator of two Pauli matrices is

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}. \tag{5.1}$$

5.3 Let $\mathbf{n}$ be any unit vector and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ be the vector whose components are the Pauli matrices. Why is it physically necessary that $\mathbf{n}\cdot\boldsymbol{\sigma}$ satisfy $(\mathbf{n}\cdot\boldsymbol{\sigma})^2 = I$, where I is the 2×2 identity matrix? Let $\mathbf{m}$ be a unit vector such that $\mathbf{m}\cdot\mathbf{n} = 0$. Why do we require that the commutator $[\mathbf{m}\cdot\boldsymbol{\sigma}, \mathbf{n}\cdot\boldsymbol{\sigma}] = 2\mathrm{i}(\mathbf{m}\times\mathbf{n})\cdot\boldsymbol{\sigma}$? Prove that that these relations follow from the algebraic properties of the Pauli matrices. You should be able to show that $[\mathbf{m}\cdot\boldsymbol{\sigma}, \mathbf{n}\cdot\boldsymbol{\sigma}] = 2\mathrm{i}(\mathbf{m}\times\mathbf{n})\cdot\boldsymbol{\sigma}$ for any two vectors $\mathbf{n}$ and $\mathbf{m}$.

5.4 Let $\mathbf{n}$ be the unit vector in the direction with polar coordinates (θ, ϕ). Write down the matrix $\mathbf{n}\cdot\boldsymbol{\sigma}$ and find its eigenvectors. Hence show that the state of a spin-half particle in which a measurement of the component of spin along $\mathbf{n}$ is certain to yield $\frac{1}{2}\hbar$ is

$$|+,\mathbf{n}\rangle = \sin(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2}|-\rangle + \cos(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2}|+\rangle, \tag{5.2}$$

where $|\pm\rangle$ are the states in which $\pm\frac{1}{2}$ is obtained when s_z is measured. Obtain the corresponding expression for $|-,\mathbf{n}\rangle$. Explain physically why the amplitudes in (5.2) have modulus $2^{-1/2}$ when $\theta = \pi/2$ and why one of the amplitudes vanishes when $\theta = \pi$.

5.5 For a spin-half particle at rest, the rotation operator $\mathbf{J}$ is equal to the spin operator $\mathbf{S}$. Use the result of Problem 5.2 to show that in this case the rotation operator $U(\boldsymbol{\alpha}) \equiv \exp(-\mathrm{i}\boldsymbol{\alpha}\cdot\mathbf{J})$ is

$$U(\boldsymbol{\alpha}) = I\cos\left(\frac{\alpha}{2}\right) - \mathrm{i}\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}\sin\left(\frac{\alpha}{2}\right), \tag{5.3}$$

where $\hat{\boldsymbol{\alpha}}$ is the unit vector parallel to $\boldsymbol{\alpha}$. Comment on the value this gives for $U(\boldsymbol{\alpha})$ when $\alpha = 2\pi$.

5.6 Justify physically the claim that the Hamiltonian of a particle that precesses in a magnetic field $\mathbf{B}$ can be written

$$H = -2\mu\mathbf{s}\cdot\mathbf{B}. \tag{5.4}$$

In a coordinate system oriented such that the z axis is parallel to $\mathbf{B}$, a proton is initially in the eigenstate $|+,x\rangle$ of s_x. Obtain expressions for the expectation values of s_x and s_y at later times. Explain the physical content of your expressions.

Bearing in mind that a rotating magnetic field must be a source of radiation, do you expect your expressions to remain valid to arbitrarily late times? What really happens in the long run?

5.7 Write down the 3×3 matrix that represents S_x for a spin-one system in the basis in which S_z is diagonal (i.e., the basis states are $|0\rangle$ and $|\pm\rangle$ with $S_z|+\rangle = |+\rangle$, etc.)

A beam of spin-one particles emerges from an oven and enters a Stern–Gerlach filter that passes only particles with $J_z = \hbar$. On exiting this filter, the beam enters a second filter that passes only particles with $J_x = \hbar$, and then finally it encounters a filter that passes only particles with $J_z = -\hbar$. What fraction of the particles stagger right through?

5.8 In the rotation spectrum of $^{12}\mathrm{C}^{16}\mathrm{O}$ the line arising from the transition $l = 4 \to 3$ is at 461.04077 GHz, while that arising from $l = 36 \to 35$ is at 4115.6055 GHz. Show from these data that in a non-rotating CO molecule the intra-nuclear distance is $s \simeq 0.113\,\mathrm{nm}$, and that the electrons provide a spring between the nuclei that has force constant $\sim 1904\,\mathrm{N\,m^{-1}}$. Hence show that the vibrational frequency of CO should lie near 6.47×10^{13} Hz (measured value is 6.43×10^{13} Hz). Hint: show from classical mechanics that the distance of O from the centre of mass is $\frac{3}{7}s$ and that the molecule's moment of inertia is $\frac{48}{7}m_\mathrm{p}s^2$. Recall also the classical relation $L = I\omega$.

5.9 Show that L_i commutes with $\mathbf{x}\cdot\mathbf{p}$ and thus also with scalar functions of $\mathbf{x}$ and $\mathbf{p}$.

5.10 The angular part of a system's wavefunction is

$$\langle\theta,\phi|\psi\rangle \propto (\sqrt{2}\cos\theta + \sin\theta\mathrm{e}^{-\mathrm{i}\phi} - \sin\theta\mathrm{e}^{\mathrm{i}\phi}).$$

What are the possible results of measurement of (a) L^2, and (b) L_z, and their probabilities? What is the expectation value of L_z?

5.11 A system's wavefunction is proportional to $\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\phi}$. What are the possible results of measurements of (a) L_z and (b) L^2?

5.12 A system's wavefunction is proportional to $\sin^2\theta$. What are the possible results of measurements of (a) L_z and (b) L^2? Give the probabilities of each possible outcome.

5.13 A box containing two spin-one gyros A and B is found to have angular-momentum quantum numbers $j=2$, $m=1$. Determine the probabilities that when J_z is measured for gyro A, the values $m=\pm1$ and 0 will be obtained.

What is the value of the Clebsch–Gordan coefficient $C(2,1;1,1,1,0)$?

5.14 The angular momentum of a hydrogen atom in its ground state is entirely due to the spins of the electron and proton. The atom is in the state $|1,0\rangle$ in which it has one unit of angular momentum but none of it is parallel to the z-axis. Express this state as a linear combination of products of the spin states $|\pm,\mathrm{e}\rangle$ and $|\pm,\mathrm{p}\rangle$ of the proton and electron. Show that the states $|x\pm,\mathrm{e}\rangle$ in which the electron has well-defined spin along the x-axis are

$$|x\pm,\mathrm{e}\rangle = \frac{1}{\sqrt{2}}\left(|+,\mathrm{e}\rangle \pm |-,\mathrm{e}\rangle\right). \tag{5.5}$$

By writing

$$|1,0\rangle = |x+,\mathrm{e}\rangle\langle x+,\mathrm{e}|1,0\rangle + |x-,\mathrm{e}\rangle\langle x-,\mathrm{e}|1,0\rangle, \tag{5.6}$$

express $|1,0\rangle$ as a linear combination of the products $|x\pm,\mathrm{e}\rangle|x\pm,\mathrm{p}\rangle$. Explain the physical significance of your result.