Introduction to Quantum Mechanics HT 2010
Problems 3 (Christmas vacation)

3.1 After choosing units in which everything, including $\hbar = 1$, the Hamiltonian of a harmonic oscillator may be written $H = \frac{1}{2}(p^2 + x^2)$, where $[x, p] = \mathrm{i}$. Show that if $|\psi\rangle$ is a ket that satisfies $H|\psi\rangle = E|\psi\rangle$, then

$$\tfrac{1}{2}(p^2 + x^2)(x \mp \mathrm{i}p)|\psi\rangle = (E \pm 1)(x \mp \mathrm{i}p)|\psi\rangle. \tag{3.1}$$

Explain how this algebra enables one to determine the energy eigenvalues of a harmonic oscillator.

3.2 Given that $A|E_n\rangle = \alpha|E_{n-1}\rangle$ and $E_n = (n + \frac{1}{2})\hbar\omega$, where the annihilation operator of the harmonic oscillator is

$$A \equiv \frac{m\omega x + \mathrm{i}p}{\sqrt{2m\hbar\omega}}, \tag{3.2}$$

show that $\alpha = \sqrt{n}$. Hint: consider $|A|E_n\rangle|^2$.

3.3 The pendulum of a grandfather clock has a period of 1 s and makes excursions of 3 cm either side of dead centre. Given that the bob weighs 0.2 kg, around what value of n would you expect its non-negligible quantum amplitudes to cluster?

3.4 Show that the minimum value of $E(p, x) \equiv p^2/2m + \frac{1}{2}m\omega^2x^2$ with respect to the real numbers p, x when they are constrained to satisfy $xp = \frac{1}{2}\hbar$, is $\frac{1}{2}\hbar\omega$. Explain the physical significance of this result.

3.5 How many nodes are there in the wavefunction $\langle x|n\rangle$ of the $n^{\rm th}$ excited state of a harmonic oscillator?

3.6 Show for a harmonic oscillator that the wavefunction of the second excited state is $\langle x|2\rangle =$ constant $\times\, (x^2/\ell^2 - 1)\mathrm{e}^{-x^2/4\ell^2}$, where $\ell \equiv \sqrt{\frac{\hbar}{2m\omega}}$, and find the normalising constant. Hint: apply $A^\dagger$ to $|0\rangle$ twice in the position representation.

3.7 Use

$$x = \sqrt{\frac{\hbar}{2m\omega}}(A + A^\dagger) = \ell(A + A^\dagger) \tag{3.3}$$

to show for a harmonic oscillator that in the energy representation the operator x is

$$x_{jk} = \ell \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots & & & & \\ \sqrt{1} & 0 & \sqrt{2} & 0 & & & & & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdots & & & & \\ & & \sqrt{3} & \cdots & & & & & \\ \cdots & & \cdots & & \cdots & & \cdots & & \\ & & & \cdots & 0 & \sqrt{n-1} & \cdots & & \\ & & & & \sqrt{n-1} & 0 & \sqrt{n} & & \\ & & & & & \sqrt{n} & 0 & \sqrt{n+1} & \cdots \\ & & & & & & \sqrt{n+1} & 0 & \\ \cdots & & \cdots & & \cdots & & \cdots & & \cdots \end{pmatrix} \tag{3.4}$$

Calculate the same entries for the matrix p_{jk}.

3.8 At $t = 0$ the state of a harmonic oscillator of mass m and frequency ω is

$$|\psi\rangle = \tfrac{1}{2}|N-1\rangle + \tfrac{1}{\sqrt{2}}|N\rangle + \tfrac{1}{2}|N+1\rangle. \tag{3.5}$$

Calculate the expectation value of x as a function of time and interpret your result physically in as much detail as you can.

3.9* In terms of the usual ladder operators A, $A^\dagger$, a Hamiltonian can be written

$$H = \mu A^\dagger A + \lambda(A + A^\dagger). \tag{3.6}$$

What restrictions on the values of the numbers μ and λ follow from the requirement for H to be Hermitian?

Show that for a suitably chosen operator B, H can be rewritten

$$H = \mu B^\dagger B + \text{constant}. \tag{3.7}$$

where $[B, B^\dagger] = 1$. Hence determine the spectrum of H.