Complex Numbers
and
Ordinary Differential Equations

Prof. J.J. Binney
Oxford University
Michaelmas Term 2002

Books: The material of this course is covered well in many texts on mathematical methods for science students, for example Boas, *Mathematical Methods in the Physical Sciences,* 2nd ed. (Wiley). A more elementary book is Stephenson, *Mathematical Methods for Science Students,* 2nd ed. (Longmans).

1 Complex Numbers

1.1 Why complex nos?

Natural numbers (positive integers) 1, 2, 3, ...

But $20 + y = 12 \quad \Rightarrow \quad y = -8 \quad \rightarrow$ integers $\quad \ldots, -3, -2, -1, 0, 1, 2, \ldots$

But $4x = 6 \quad \Rightarrow \quad x = \frac{3}{2} \quad \rightarrow$ rationals

But $x^2 = 2 \quad \Rightarrow \quad x = \sqrt{2} \quad \rightarrow$ irrationals

But $x^2 = -1 \quad \Rightarrow \quad x = \mathrm{i} \quad \rightarrow$ complex nos

Multiples of i are called **pure imaginary** numbers. A general complex number is the sum of a multiple of 1 and a multiple of i such as $z = 2 + 3\mathrm{i}$. We often use the notation $z = a + \mathrm{i}b$, where a and b are real. We define operators for extracting a, b from z: $a \equiv \Re\mathrm{e}(z),\ b \equiv \Im\mathrm{m}(z)$. We call a the **real part** and b the **imaginary part** of z.

These rules allow us to add and multiply complex numbers:

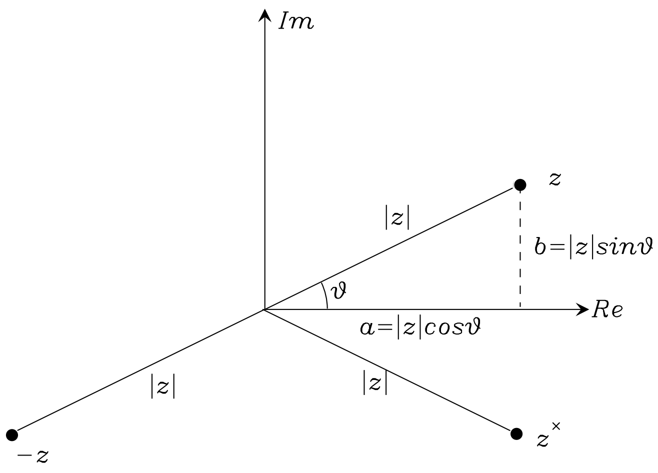
$$
\begin{aligned}
z_1 \pm z_2 &\equiv (a_1 \pm a_2) + \mathrm{i}(b_1 \pm b_2) \\
z_1 z_2 &= (a_1 + \mathrm{i}b_1)(a_2 + \mathrm{i}b_2) \\
&\equiv (a_1 a_2 - b_1 b_2) + \mathrm{i}(a_1 b_2 + b_1 a_2)
\end{aligned} \tag{1.1}
$$

It is nice to define division also. We First define $z^* \equiv a - \mathrm{i}b$, the **complex conjugate** of z, and note that $|z|^2 \equiv zz^* = (a^2 + b^2)$ is real (and > 0). So we can define

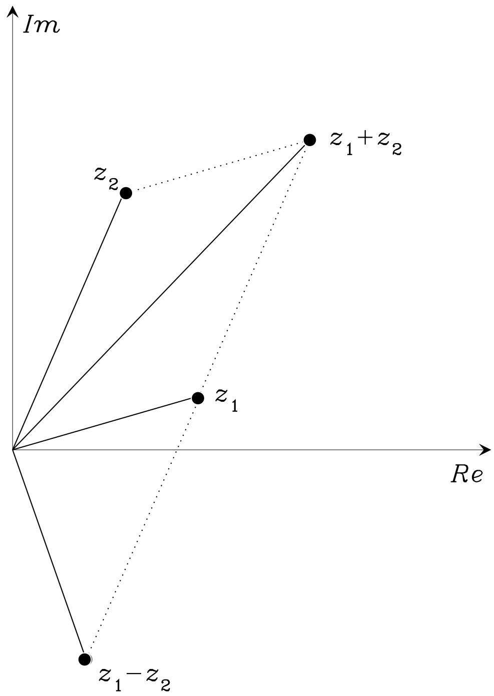
$$
\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} \tag{1.2}
$$

1.2 Argand diagram (complex plane)

Each $z \rightarrow$ point (a, b) in plane:



Let $\arg(z) \equiv \theta = \arctan(b/a)$. Then $z = |z|(\cos\theta + \mathrm{i}\sin\theta)$



1.3 Simple functions of z and de Moivre's Theorem

Have series

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \end{aligned}$$

Define

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots \tag{1.3}$$

Special case $z = i\theta$

$$\begin{aligned} e^{i\theta} &= \Big(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\Big) + i\Big(\theta - \frac{\theta^3}{3!} + \cdots\Big) \\ &= \underset{(a)}{\cos\theta} + i\underset{(b)}{\sin\theta} \qquad (\textit{de Moivre's theorem}) \end{aligned} \tag{1.4}$$

Thus

$$\begin{aligned} z &= |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta} \\ z^* &= |z|(\cos\theta - i\sin\theta) = |z|e^{-i\theta} \\ \frac{1}{z} &= \frac{z^*}{zz^*} = \frac{e^{-i\theta}}{|z|}. \end{aligned} \tag{1.5}$$

Adding and then subtracting the first two of equations (1.5) we find that

$$\begin{aligned} \cos\theta &= \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ \sin\theta &= \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}) \end{aligned} \tag{1.6}$$

Also

$$\begin{aligned} e^{\ln z} = z = |z|e^{i\theta} &= e^{\ln|z|}e^{i\theta} = e^{\ln|z| + i\theta} \\ \Rightarrow \quad \ln z &= \underset{(a)}{\ln|z|} + i\,\underset{(b)}{\arg(z)} \end{aligned} \tag{1.7}$$

1.3.1 Trigonometric identities Have

$$\begin{aligned}\cos(a+b)+\mathrm{i}\sin(a+b) &= \mathrm{e}^{\mathrm{i}(a+b)} = \mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}b}\\ &= (\cos a+\mathrm{i}\sin a)(\cos b+\mathrm{i}\sin b)\\ &= (\cos a\cos b-\sin a\sin b)+\mathrm{i}(\cos a\sin b+\sin a\cos b)\end{aligned}$$

Comparing real and imaginary parts on the two sides, we deduce that

$$\begin{aligned}\cos(a+b) &= \cos a\cos b-\sin a\sin b\\ \sin(a+b) &= \sin a\cos b+\cos a\sin b\end{aligned} \tag{1.8}$$

We use the last result to evaluate the cosine of a complex number:

$$\cos z=\cos(a+\mathrm{i}b)=\cos a\cos(\mathrm{i}b)-\sin a\sin(\mathrm{i}b). \tag{1.9}$$

Now from (1.6) the cosine and the sine of an imaginary angle are

$$\begin{aligned}\cos(\mathrm{i}b) &= \tfrac{1}{2}(\mathrm{e}^{-b}+\mathrm{e}^{b})=\cosh b\\ \sin(\mathrm{i}b) &= \tfrac{1}{2\mathrm{i}}(\mathrm{e}^{-b}-\mathrm{e}^{b})=\mathrm{i}\sinh b,\end{aligned} \tag{1.10}$$

where we have used the definitions of the hyperbolic functions

$$\begin{aligned}\cosh b &\equiv \tfrac{1}{2}(\mathrm{e}^{b}+\mathrm{e}^{-b})\\ \sinh b &\equiv \tfrac{1}{2}(\mathrm{e}^{b}-\mathrm{e}^{-b}).\end{aligned} \tag{1.11}$$

Substituting from (1.10) in (1.9) we have

$$\cos z=\cos a\cosh b-\mathrm{i}\sin a\sinh b. \tag{1.12}$$

Analogously

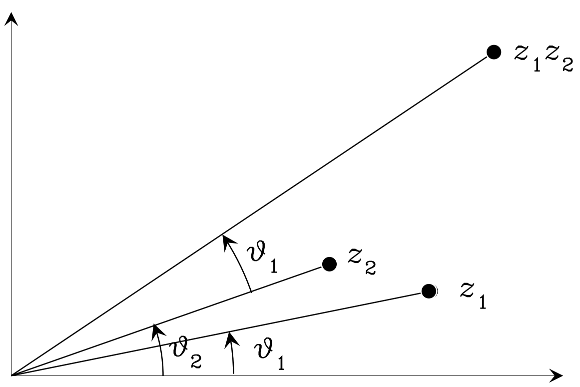
$$\sin z=\sin a\cosh b+\mathrm{i}\cos a\sinh b. \tag{1.13}$$

Note:

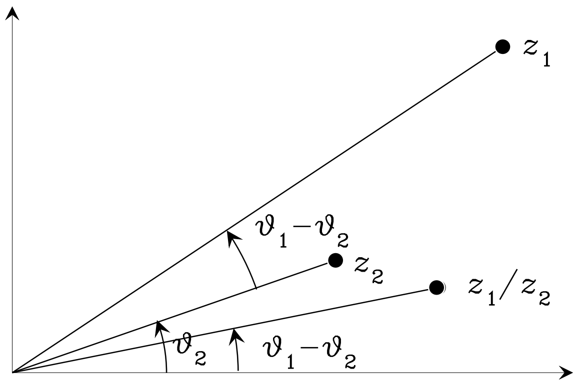
Hyperbolic functions get their name from the identity $\cosh^2\theta-\sinh^2\theta=1$, which is readily proved from (1.11) and is reminiscent of the equation of a hyperbola, $x^2-y^2=1$.

1.3.2 Graphical representation of multiplication & division

$$z_1z_2=|z_1||z_2|\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}$$



$$\frac{z_1}{z_2}=\frac{|z_1|}{|z_2|}\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)}$$



Example 1.1

Find the modulus $|z_1/z_2|$ when $\begin{cases} z_1 = 1 + 2\mathrm{i} \\ z_2 = 1 - 3\mathrm{i} \end{cases}$

Clumsy method:

$$\begin{aligned} \left|\frac{z_1}{z_2}\right| &= \left|\frac{1+2\mathrm{i}}{1-3\mathrm{i}}\right| = \frac{|z_1 z_2^*|}{|z_2|^2} \\ &= \frac{|(1+2\mathrm{i})(1+3\mathrm{i})|}{1+9} = \frac{|(1-6)+\mathrm{i}(2+3)|}{10} \\ &= \frac{\sqrt{25+25}}{10} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \end{aligned}$$

Elegant method:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{1+4}}{\sqrt{1+9}} = \frac{1}{\sqrt{2}}$$

1.3.3 Summing series with de Moivre

Example 1.2

Prove that for $0 < r < 1$

$$\sum_{n=0}^{\infty} r^n \sin(2n+1)\theta = \frac{(1+r)\sin\theta}{1 - 2r\cos 2\theta + r^2}$$

Proof:

$$\begin{aligned} \sum_{n=0}^{\infty} r^n \sin(2n+1)\theta &= \sum_n r^n \Im\mathrm{m}(\mathrm{e}^{\mathrm{i}(2n+1)\theta}) = \Im\mathrm{m}\Big(\mathrm{e}^{\mathrm{i}\theta} \sum_n (r\mathrm{e}^{2\mathrm{i}\theta})^n\Big) \\ &= \Im\mathrm{m}\left(\mathrm{e}^{\mathrm{i}\theta}\frac{1}{1 - r\mathrm{e}^{2\mathrm{i}\theta}}\right) \\ &= \Im\mathrm{m}\left(\frac{\mathrm{e}^{\mathrm{i}\theta}(1 - r\mathrm{e}^{-2\mathrm{i}\theta})}{(1 - r\mathrm{e}^{2\mathrm{i}\theta})(1 - r\mathrm{e}^{-2\mathrm{i}\theta})}\right) \\ &= \frac{\sin\theta + r\sin\theta}{1 - 2r\cos 2\theta + r^2} \end{aligned}$$

1.4 Curves in the complex plane

Example 1.3

What is the locus in the Argand diagram that is defined by $\left|\frac{z-\mathrm{i}}{z+\mathrm{i}}\right| = 1$?

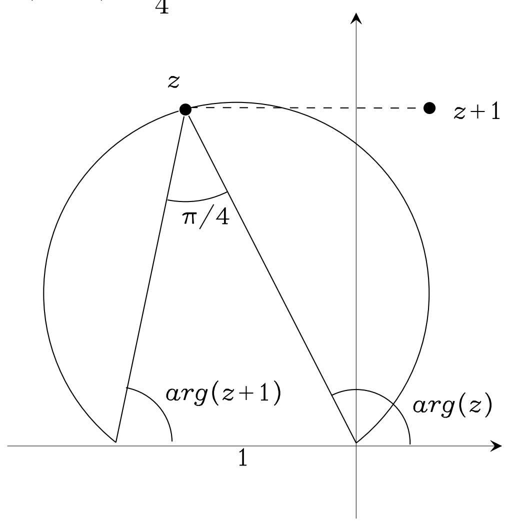
Equivalently $\frac{|z-\mathrm{i}|}{|z+\mathrm{i}|} = 1$, so distance from $(0,1)$ same as distance from $(0,-1)$

Hence solution is "real axis"

Example 1.4

What is the locus in the Argand diagram that is defined by $\arg\left(\frac{z}{z+1}\right) = \frac{\pi}{4}$?

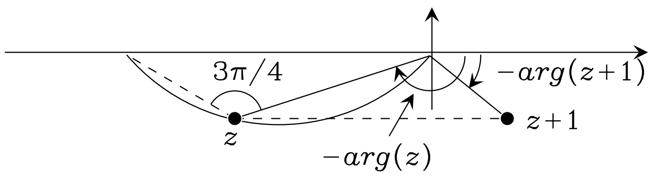
Equivalently $\arg(z) - \arg(z+1) = \frac{\pi}{4}$



Solution: "portion of circle through $(0,0)$ and $(-1,0)$"

The x-coordinate of the centre is $-\frac{1}{2}$ by symmetry. The angle subtended by a chord at the centre is twice that subtended at the circumference, so here it is $\pi/2$. With this fact it easily follows that the y-coordinate of the centre is $\frac{1}{2}$.

The lower portion of circle is $\arg\left(\frac{z}{z+1}\right) = -\frac{3\pi}{4}$



1.5 Roots of polynomials

Complex numbers enable us to find roots for any polynomial

$$P(z) \equiv a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0. \tag{1.14}$$

That is, there is at least one, and perhaps as many as n complex numbers z_i such that $P(z_i) = 0$. Many physical problems involve such roots.

In the case $n = 2$ you already know a general formula for the roots. There is a similar formula for the case $n = 3$ and historically this is important because it lead to the invention of complex numbers. About 1515, Scipione del Ferro at Balogna found a formula for the roots of

$$x^3 + 3px = 2q, \tag{1.15}$$

but he kept his formula secret. In 1535 Tartaglia, 34 years younger than del Ferro, claimed to have discovered a formula for the solution of $x^3 + rx^2 = 2q$.† Del Ferro didn't believe him and challenged him to an equation-solving match. Galvanized, Tartaglia added (1.15) to his repertoire before the match and won hands down. But he too kept his formulae secret until Cardano got the formula for (1.15) from either del Ferro or Tartaglia and published it in his book *Ars Magna* in 1545. Here's the derivation. We have

$$(a-b)^3 + 3ab(a-b) = a^3 - b^3. \tag{1.16}$$

If we can choose a and b such that

$$3ab = 3p \quad \text{and} \quad a^3 - b^3 = 2q, \tag{1.17}$$

then (1.16) becomes the cubic (1.15) with $x = (a-b)$, which we will know. Solving the simultaneous equations (1.17) for a by elimination of b we get a quadratic equation in a^3

$$a^6 - 2qa^3 - p^3 = 0. \tag{1.18}$$

Solving this and the equivalent equation $(q \to -q)$ for b we find*

$$x = a - b = \left[q + (q^2+p^3)^{1/2}\right]^{1/3} - \left[-q + (q^2+p^3)^{1/2}\right]^{1/3}. \tag{1.19}$$

By considering the existence of stationary points it is easy to show that a cubic with three real roots has $q^2 + p^3 < 0$. The Cardano–Tartaglia formula (1.19) gives these as the differences of complex numbers. For this reason the *Ars Magna* contained the elements of the theory of complex numbers.

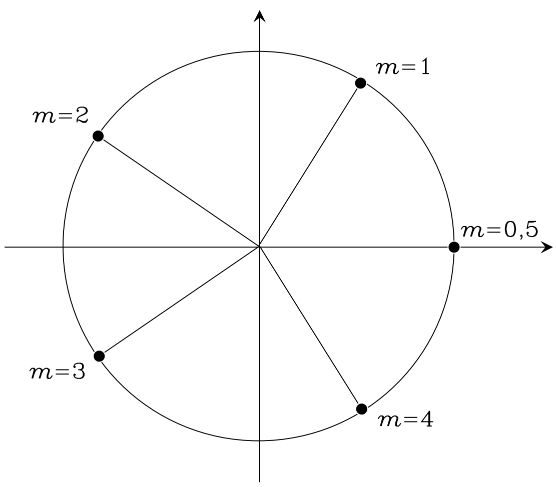
The *Ars Magna* showed how the general quartic equation can be reduced to a cubic equation, and hence gave a formula for the roots in terms of radicals of the coefficients in the original equation. Interestingly, it can be shown that such formulae do not exist for equations of higher order, such as quintics.

1.5.1 Special polyniomials We can, however, find the roots of specially simple polynomials. We start with something really trivial: $x^n = 1 \quad \Rightarrow \quad x = 1^{1/n}$

$$1 = \mathrm{e}^{2m\pi \mathrm{i}} \quad \Rightarrow \quad 1^{1/n} = \mathrm{e}^{2m\pi\mathrm{i}/n} = \cos\left(\frac{2m\pi}{n}\right) + \mathrm{i}\sin\left(\frac{2m\pi}{n}\right)$$

e.g.

$$1^{1/5} = \cos\left(\frac{2m\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2m\pi}{5}\right) \quad (m = 0,1,2,3,4).$$



† By making the substitution $x \equiv x' - k$ we can turn the general cubic $x^3 + a_2x^2 + a_1x + a_0 = 0$ into either del Ferro's form or Tartaglia's form.

* When solving for a and b, the second of equations (1.17) requires us to choose the same sign for the square root.

Note:

We shall often need the coefficients of $x^r y^{n-r}$ in $(x+y)^n$. These are conveniently obtained from **Pascal's triangle**:

$$\begin{array}{lccccccccccc}
(x+y)^0 & & & & & & 1 & & & & & \\
(x+y)^1 & & & & & 1 & & 1 & & & & \\
(x+y)^2 & & & & 1 & & 2 & & 1 & & & \\
(x+y)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(x+y)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(x+y)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Each row is obtained from the one above by adding the numbers to right and left of the position to be filled in.

Example 1.5

Consider the equation $(z+\mathrm{i})^7 + (z-\mathrm{i})^7 = 0$.

$$\begin{aligned}
&\Rightarrow \quad z^7 - 21z^5 + 35z^3 - 7z = 0 \\
&\Rightarrow \quad z^6 - 21z^4 + 35z^2 - 7 = 0 \\
&\Rightarrow \quad w^3 - 21w^2 + 35w - 7 = 0 \quad (w \equiv z^2)
\end{aligned}$$

We also have

$$\left(\frac{z+\mathrm{i}}{z-\mathrm{i}}\right)^7 = -1 = \mathrm{e}^{(2m+1)\pi\mathrm{i}}$$

$$\Rightarrow \quad \frac{z+\mathrm{i}}{z-\mathrm{i}} = \mathrm{e}^{(2m+1)\pi\mathrm{i}/7} \quad \Rightarrow \quad z\left(1-\mathrm{e}^{(2m+1)\pi\mathrm{i}/7}\right) = -\mathrm{i}\left(1+\mathrm{e}^{(2m+1)\pi\mathrm{i}/7}\right)$$

$$\Rightarrow \quad z = \mathrm{i}\frac{\mathrm{e}^{(2m+1)\pi\mathrm{i}/7}+1}{\mathrm{e}^{(2m+1)\pi\mathrm{i}/7}-1} = \mathrm{i}\frac{\mathrm{e}^{(2m+1)\pi\mathrm{i}/14}+\mathrm{e}^{-(2m+1)\pi\mathrm{i}/14}}{\mathrm{e}^{(2m+1)\pi\mathrm{i}/14}-\mathrm{e}^{-(2m+1)\pi\mathrm{i}/14}} = \mathrm{i}\frac{2\cos\left(\frac{2m+1}{14}\pi\right)}{2\mathrm{i}\sin\left(\frac{2m+1}{14}\pi\right)}$$

Thus the roots of $w^3 - 21w^2 + 35w - 7 = 0$ are $w = \cot^2\left(\frac{2m+1}{14}\pi\right) \quad (m = 0, 1, 2)$.

Sometimes the underlying equation is not obvious.

Example 1.6

Find the roots of

$$z^3 + 7z^2 + 7z + 1 = 0.$$

Ninth row of Pascal's triangle is

$$1\ 8\ 28\ 56\ 70\ 56\ 28\ 8\ 1,$$

so

$$\begin{aligned}
\tfrac{1}{2}[(z+1)^8 - (z-1)^8] &= 8z^7 + 56z^5 + 56z^3 + 8z \\
&= 8z[w^3 + 7w^2 + 7w + 1] \quad (w \equiv z^2).
\end{aligned}$$

Now $(z+1)^8 - (z-1)^8 = 0$ when $\dfrac{z+1}{z-1} = \mathrm{e}^{2m\pi i/8}$, i.e. when

$$z = \frac{\mathrm{e}^{m\pi\mathrm{i}/4}+1}{\mathrm{e}^{m\pi\mathrm{i}/4}-1} = -\mathrm{i}\cot(m\pi/8) \quad (m = 1, 2, \ldots, 7),$$

so the roots of the given equation are $z = -\cot^2(m\pi/8)$, $m = 1, 2, 3$.

1.5.2 Characterizing a polynomial by its roots Knowledge of a polynomial's roots enables us to express the polynomial as a product of linear terms

$$\begin{aligned} a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 &= a_n(z-r_1)(z-r_2)\cdots(z-r_n) \\ &= a_n\Big(z^n - z^{n-1}\sum_{j=1}^{n} r_j + \cdots + (-1)^n \prod_{j=1}^{n} r_j\Big). \end{aligned}$$

Comparing the coefficients of z^{n-1} and z^0 on the two sides, we deduce that

$$\frac{a_{n-1}}{a_n} = -\sum_j r_j \quad ; \quad \frac{a_0}{a_n} = (-1)^n \prod_j r_j \tag{1.20}$$

Example 1.7

Show that $\sum_{m=0}^{2} \cot^2\left(\frac{2m+1}{14}\pi\right) = 21$

Solution: From Example 1.5 we have that these numbers are the roots of $w^3 - 21w^2 + 35w - 7 = 0$.

A polynomial may be characterized by (i) its roots and (ii) any a_n.

Example 1.8

Show that

$$\frac{z^{2m}-a^{2m}}{z^2-a^2} = \Big(z^2 - 2az\cos\frac{\pi}{m} + a^2\Big)\Big(z^2 - 2az\cos\frac{2\pi}{m} + a^2\Big)\cdots\Big(z^2 - 2az\cos\frac{(m-1)\pi}{m} + a^2\Big).$$

Solution: Consider $P(z) \equiv z^{2m} - a^{2m}$, a polynomial of order $2m$ with leading term $a_{2m} = 1$ and roots $z_r = a\mathrm{e}^{r\pi i/m}$. Define

$$Q(z) \equiv (z^2 - a^2)\Big(z^2 - 2az\cos\frac{\pi}{m} + a^2\Big)\Big(z^2 - 2az\cos\frac{2\pi}{m} + a^2\Big)\cdots\Big(z^2 - 2az\cos\frac{(m-1)\pi}{m} + a^2\Big).$$

This polynomial is of order $2m$ with leading coeff. $a_{2m} = 1$ and with roots that are the numbers

$$\begin{aligned} z_r &= a\cos\frac{r\pi}{m} \pm \sqrt{a^2\cos^2\frac{r\pi}{m} - a^2} \\ &= a\Big(\cos\frac{r\pi}{m} \pm \mathrm{i}\sqrt{1-\cos^2\frac{r\pi}{m}}\Big) = a\mathrm{e}^{\pm \mathrm{i}r\pi/m} \quad (r = 0, 1, \ldots, m). \end{aligned}$$

Thus P and Q are identical.

2 Linear Differential Equations

A **differential equation** is an equation in which an expression involving derivatives of an unknown function are set equal to a known function. For example

$$\frac{\mathrm{d}f}{\mathrm{d}x} + xf = \sin x \tag{2.1}$$

is a differential equation for $f(x)$. To determine a unique solution of a differential equation we require some initial data; in the case of (2.1), the value of f at some point x. These data are often called **initial conditions**. Below we'll discuss how many initial conditions one typically needs.

Perhaps Newton's most brilliant insight was that differential equations enable us to encapsulate physical laws: the equation governs events everywhere and at all times; the rich variety of experience arises because at different places and times different initial conditions select different solutions. Since differential equations are of such transcending importance for physics, let's talk about them in some generality.

2.1 Differential operators

Every differential equation involves a **differential operator**.

functions: numbers $\to$ numbers (e.g. $x \to e^x$)

operators: functions $\to$ functions (e.g. $f \to \alpha f;\ f \to 1/f;\ f \to f + \alpha; \ldots$)

A differential operator does this mapping by differentiating the function one or more times (and perhaps adding in a function, or multiplying by one, etc).

$$\left(\text{e.g. } f \to \frac{\mathrm{d}f}{\mathrm{d}x};\ f \to \frac{\mathrm{d}^2 f}{\mathrm{d}x^2};\ f \to 2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f\frac{\mathrm{d}f}{\mathrm{d}x}; \ldots\right)$$

2.1.1 Order of a differential operator The **order** of a differential operator is the order of the highest derivative contained in it. So

$$\begin{aligned} L(f) &\equiv \frac{\mathrm{d}f}{\mathrm{d}x} + 3f \quad \text{is first order,} \\ L(f) &\equiv \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 3f \quad \text{is second order,} \\ L(f) &\equiv \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 4\frac{\mathrm{d}f}{\mathrm{d}x} \quad \text{is second order.} \end{aligned}$$

2.1.2 Linear operators L is a **linear operator** iff

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g), \tag{2.2}$$

where α and β are (possibly complex) numbers.

$$\left(\text{e.g. } f \to \frac{\mathrm{d}f}{\mathrm{d}x} \text{ and } f \to \alpha f \text{ are linear, but } f \to \frac{1}{f} \text{ and } f \to f + \alpha \text{ are not.}\right)$$

Note:

An expression of the type $\alpha f + \beta g$ that is a sum of multiples of two or more functions is called a **linear combination** of the functions.

To a good approximation, many physical systems are described by linear differential equations $L(f) = 0$. Classical electrodynamics provides a notable example: the equations (Maxwell's) governing electric and magnetic fields in a vacuum are linear. The related equation governing the generation of a Newtonian gravitational field is also linear.

Suppose f and g are two solutions of the linear equation $L(y) = 0$ for different initial conditions. For example, if L symbolizes Maxwell's equations, f and g might describe the electric fields generated by different distributions of charges. Then since L is linear, $L(f+g) = 0$, so $(f+g)$ describes the electric field generated by both charge distributions taken together. This idea, that if the governing equations are linear, then the response to two stimuli taken together is just the sum of the responses to the stimuli taken separately, is known as the **principle of superposition**. This principle is widely used to find the required solution to linear differential equations: we start by finding some very simple solutions that individually don't satisfy our initial conditions and then we look for linear combinations of them that do.

Linearity is almost always an approximation that breaks down if the stimuli are very large. For example, in consequence of the linearity of Maxwell's equations, the beam from one torch will pass right through the beam of another torch without being affected by it. But the beam from an extremely strong source of light would scatter a torch beam because the vacuum contains 'virtual' electron-positron pairs which respond non-negligibly to the field of a powerful beam, and the excited electro-positron pairs can then scatter the torch beam. In a similar way, light propagating through a crystal (which is full of positive and negative charges) can rather easily modify the electrical properties of a crystal in a way that affects a second light beam – this is the idea behind non-linear optics, now an enormously important area technologically. Gravity too is non-linear for very strong fields.

While non-linearity is the generic case, the regime of weak stimuli in which physics is to a good approximation linear is often a large and practically important one. Moreover, when we do understand non-linear processes quantitatively, this is often done using concepts that arise in the linear regime. For example, any elementary particle, such as an electron or a quark, is a weak-field, linear-response construct of quantum field theory.

2.1.3 Arbitrary constants & general solutions How many initial conditions do we need to specify to pick out a unique solution of $L(f) = 0$? Arrange $Lf \equiv a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0 = 0$ as

$$f^{(n)}(x) = -\left(\frac{a_{n-1}}{a_n} f^{(n-1)}(x) + \cdots + \frac{a_0}{a_n} f\right). \tag{2.3}$$

If we differentiate both sides of this equation with respect to x, we obtain an expression for $f^{(n+1)}(x)$ in terms of $f^{(n)}(x)$ and lower derivatives. With the help of (2.3) we can eliminate $f^{(n)}(x)$ from this new equation, and thus obtain an expression for $f^{(n+1)}(x)$ in terms of $f(x)$ and derivatives up to $f^{(n-1)}(x)$. By differentiating both sides of our new equation and again using (2.3) to eliminate $f^{(n)}$ from the resulting equation, we can obtain an expression for $f^{(n+2)}(x)$ in terms of $f(x)$ and derivatives up to $f^{(n-1)}(x)$. Repeating this procedure a sufficient number of times we can obtain an expression for *any* derivative of f in terms of $f(x)$ and derivatives up to $f^{(n-1)}$. Consequently, if the values of these n functions are given at any point x_0 we can evaluate the Taylor series

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \tfrac{1}{2}(x - x_0)^2 f''(x_0) + \cdots \tag{2.4}$$

for any value of x that lies near enough to x_0 for the series to converge. Consequently, the functional form of $f(x)$ is determined by the original n^{th} order differential equation and the n initial conditions $f(x_0), \ldots, f^{(n-1)}(x_0)$. Said another way, to pick out a unique solution to an nth order equation, we need n initial conditions.

The **general solution** of a differential equation is one that contains a sufficient supply of arbitrary constants to allow it to become *any* solution of the equation if these constants are assigned appropriate values. We have seen that once the n numbers $f^{(r)}(x_0)$ for $r = 0, \ldots, n-1$ have been specified, the solution to the linear nth-order equation $Lf = 0$ is uniquely determined. This fact suggests that the general solution of $Lf = 0$ should include n arbitrary constants, one for each derivative. This is true, although the constants don't have to be the values of individual derivatives; all that is required is that appropriate choices of the constants will cause the rth derivative of the general solution to adopt any specified value.

Given the general solution we can construct n particular solutions $f_1, \ldots, f_n$ as follows: let f_1 be the solution in which the first arbitrary constant, k_1, is unity and the others zero, f_2 be the solution in which the second constant, k_2, is unity and the other zero, etc. It is easy to see that the general solution is

$$f(x) = \sum_{r=1}^{n} k_r f_r(x). \tag{2.5}$$

That is, the general solution is a linear combination of n **particular solutions**, that is, solutions with no arbitrary constant.

2.2 Inhomogeneous terms

We've so far imagined the stimuli to be encoded in the initial conditions. It is sometimes convenient to formulate a physical problem so that at least some of the stimuli are encoded by a function that we set our differential operator equal to. Thus we write

$$\underset{\substack{\text{given}\\ \text{homogeneous}}}{L}\ \underset{\text{sought}}{(f)} = \underset{\substack{\text{given}\\ \text{inhomogeneous}}}{h(x)} \tag{2.6}$$

Suppose f_1 is the general solution of $Lf = 0$ and f_0 is any solution of $Lf = h$. We call f_1 the **complementary function** and f_0 the **particular integral** and have that then general solution of $Lf = h$ is

$$\underset{\text{Complementary fn}}{f_1} + \underset{\text{Particular integral}}{f_0}. \tag{2.7}$$

2.3 First-order linear equations

Any first-order linear equation can be written in the form

$$\frac{\mathrm{d}f}{\mathrm{d}x} + q(x)f = h(x). \tag{2.8}$$

The general solution will have one arbitrary constant. It can be found by seeking a function $I(x)$ such that

$$I\frac{\mathrm{d}f}{\mathrm{d}x} + Iqf = \frac{\mathrm{d}If}{\mathrm{d}x} = Ih \quad \Rightarrow \quad f(x) = \frac{1}{I(x)}\int_{x_0}^{x} I(x')h(x')\mathrm{d}x'. \tag{2.9}$$

x_0 is the required arbitrary constant in the solution, and I is called the **integrating factor**. We need $Iq = \mathrm{d}I/\mathrm{d}x$, so

$$\ln I = \int q\,\mathrm{d}x \quad \Rightarrow \quad I = \mathrm{e}^{\int q\,\mathrm{d}x}. \tag{2.10}$$

Example 2.1

Solve

$$2x\frac{\mathrm{d}f}{\mathrm{d}x} - f = x^2.$$

In standard form the equation reads

$$\frac{\mathrm{d}f}{\mathrm{d}x} - \frac{f}{2x} = \tfrac{1}{2}x$$

so $q = -\dfrac{1}{2x}$ and by (2.10) $I = \mathrm{e}^{-\frac{1}{2}\ln x} = \dfrac{1}{\sqrt{x}}$.

Plugging this into (2.9) we have $f = \frac{1}{2}\sqrt{x}\int_{x_0}^{x}\sqrt{x'}\,\mathrm{d}x' = \frac{1}{3}(x^2 - x_0^{3/2}x^{1/2})$.

2.4 Second-order linear equations*

The general second-order linear equation can be written in the form

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}f}{\mathrm{d}x} + q(x)f = h(x). \tag{2.11}$$

Is there an integrating factor? Suppose $\exists\ I(x)$ s.t. $\dfrac{\mathrm{d}^2 If}{\mathrm{d}x^2} = Ih$. Then

$$2\frac{\mathrm{d}I}{\mathrm{d}x} = Ip \quad \text{and} \quad \frac{\mathrm{d}^2 I}{\mathrm{d}x^2} = Iq. \tag{2.12}$$

These equations are unfortunately incompatible in most cases. Thus we cannot count on there being an integrating factor.

Now suppose we have a solution u:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}u}{\mathrm{d}x} + q(x)u = 0, \tag{2.13}$$

Then write $f = uv$ and $u' \equiv \dfrac{\mathrm{d}u}{\mathrm{d}x}$ etc. so that

$$f' = u'v + uv' \quad ; \quad f'' = u''v + 2u'v' + uv''. \tag{2.14}$$

Substituting these results into (2.11) we obtain

$$\begin{aligned} h &= f'' + pf' + qf \\ &= u''v + 2u'v' + uv'' + pu'v + puv' + quv \\ &= v(u'' + pu' + qu) + uv'' + 2u'v' + puv' \\ &= \qquad 0 \qquad\quad + uv'' + 2u'v' + puv'. \end{aligned} \tag{2.15}$$

Now define $w \equiv v'$ and find

$$uw' + (2u' + pu)w = h \quad \Rightarrow \quad \begin{cases} \mathrm{IF} = \exp\left[\int \left(2\frac{u'}{u} + p\right)\mathrm{d}x\right] \\ \quad = u^2 \mathrm{e}^{\int p\,\mathrm{d}x}. \end{cases} \tag{2.16}$$

Finally integrate

$$v'(x) = w(x) = u^{-2}(x)\mathrm{e}^{-\int^x p\,\mathrm{d}x} \int_{x_0}^{x} \mathrm{e}^{\int^{x'} p\,\mathrm{d}x} hu^2\,\mathrm{d}x'. \tag{2.17}$$

Thus if we can find one solution, u, of any second-order linear equation, we can find the general solution $f(x) = \alpha u(x) + u(x)v(x, x_0)$. Unfortunately, there is no general method for finding the first solution. So let's restrict attention to second-order equations for which this *can* be done.

* Lies beyond the syllabus

2.5 Equations with constant coefficients

Suppose the coefficients of the unknown function f and its derivatives are mere constants:

$$Lf = a_2\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}f}{\mathrm{d}x} + a_0 f = h(x). \tag{2.18}$$

We look for a complementary function $y(x)$ that satisfies $Ly = 0$. We try $y = \mathrm{e}^{\alpha x}$. Substituting this into $a_2 y'' + a_1 y' + a_0 y = 0$ we find that the equation is satisfied $\forall x$ provided

$$a_2\alpha^2 + a_1\alpha + a_0 = 0. \tag{2.19}$$

This condition for the exponent α is called the **auxilliary equation**. It has two roots

$$\alpha_\pm \equiv \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}, \tag{2.20}$$

so the CF is

$$y = A_+\mathrm{e}^{\alpha_+ x} + A_-\mathrm{e}^{\alpha_- x}. \tag{2.21}$$

Example 2.2

Solve

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 0.$$

The auxilliary equation is $(\alpha+3)(\alpha+1) = 0$, so the CF is $y = A\mathrm{e}^{-3x} + B\mathrm{e}^{-x}$.

Example 2.3

Solve

$$Ly = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0.$$

The auxilliary equation is $\alpha = \frac{1}{2}(2 \pm \sqrt{4-20}) = 1 \pm 2\mathrm{i}$, so $y = A\mathrm{e}^{(1+2\mathrm{i})x} + B\mathrm{e}^{(1-2\mathrm{i})x}$. But this is complex!

However, L is real operator. So $0 = \Re\mathrm{e}(Ly) = L[\Re\mathrm{e}(y)]$ and $\Re\mathrm{e}(y)$ is also a solution. Ditto $\Im\mathrm{m}(y)$. Consequently the solution can be written

$$y = \mathrm{e}^x\big[A'\cos(2x) + B'\sin(2x)\big].$$

Example 2.4

Find the solutions to the equation of Exercise 2.3 for which $y(0) = 1$ and $(\mathrm{d}y/\mathrm{d}x)_0 = 0$.

Solution: We obtain simultaneous equations for A' and B' by evaluating the general solution and its derivative at $x = 0$:

$$\begin{aligned}1 &= A'\\ 0 &= A' + 2B'\end{aligned} \quad\Rightarrow\quad B' = -\tfrac{1}{2} \quad\Rightarrow\quad y = \mathrm{e}^x\big[\cos(2x) - \tfrac{1}{2}\sin(2x)\big].$$

2.5.1 Factorization of operators & repeated roots The auxilliary equation (2.19) is just the differential equation $Lf = 0$ with $\mathrm{d}/\mathrm{d}x$ replaced by α. So just as the roots of a polynomial enables us to express the polynomial as a product of terms linear in the variable, so the knowledge of the roots of the auxilliary equation allows us to express L as a product of two first-order differential operators:

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_-\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_+\right)f &= \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - (\alpha_- + \alpha_+)\frac{\mathrm{d}f}{\mathrm{d}x} + \alpha_-\alpha_+ f\\ &= \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{a_1}{a_2}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{a_0}{a_2} \equiv \frac{Lf}{a_2},\end{aligned} \tag{2.22}$$

where have used our formulae (1.20) for the sum and product of the roots of a plynomial. The CF is made up of exponentials because

$$\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_-\Big)\mathrm{e}^{\alpha_- x} = 0 \quad ; \quad \Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_+\Big)\mathrm{e}^{\alpha_+ x} = 0.$$

What happens if $a_1^2 - 4a_2a_0 = 0$? Then $\alpha_- = \alpha_+ = \alpha$ and

$$Lf = \Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\Big)f. \tag{2.23}$$

It follows that

$$\begin{aligned} L\big(x\mathrm{e}^{\alpha x}\big) &= \Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\Big)x\mathrm{e}^{\alpha x} \\ &= \Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\Big)\mathrm{e}^{\alpha x} = 0, \end{aligned}$$

and the CF is $y = A\mathrm{e}^{\alpha x} + Bx\mathrm{e}^{\alpha x}$.

Example 2.5
Solve

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

The auxilliary equation is $(\alpha - 1)^2 = 0$, so $y = A\mathrm{e}^x + Bx\mathrm{e}^x$.

2.5.2 Equations of higher order These results we have just derived generalize easily to linear equations with constant coeffs of any order.

Example 2.6
Solve

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - 2\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

The auxilliary equation is $(\alpha - 1)^2(\alpha - \mathrm{i})(\alpha + \mathrm{i}) = 0$, so

$$y = \mathrm{e}^x(A + Bx) + C\cos x + D\sin x.$$

2.6 Particular integrals

Recall the general solution of $Lf = h$ is CF $+ f_0$ where the particular integal f_0 is *any* function for which $Lf_0 = h$. There is a general technique for finding PIs. This technique, which centres on **Green's functions**, lies beyond the syllabus although it is outlined in Chapter 6. For simple inhomogeneous part h we can get by with the use of trial functions. The type of function to be tried depends on the nature of h.

2.6.1 Polynomial h Suppose h is a sum of some powers of x,

$$h(x) = b_0 + b_1x + b_2x^2 + \cdots \tag{2.24}$$

Then we try

$$\begin{aligned} f(x) &= c_0 + c_1x + c_2x^2 + \cdots \\ \Rightarrow \quad f' &= c_1 + 2c_2x + \cdots \\ f'' &= 2c_2 + \cdots \end{aligned} \tag{2.25}$$

so

$$\begin{aligned} h(x) = a_2f'' + a_1f' + a_0f =& (a_0c_0 + a_1c_1 + a_22c_2 + \cdots) \\ &+ (a_0c_1 + a_12c_2 + \cdots)x \\ &+ (a_0c_2 + \cdots)x^2 \\ &+ \cdots \end{aligned} \tag{2.26}$$

Comparing powers of x^0, $x^1, \ldots$ on the two sides of this equation, we obtained coupled linear equations for the c_r in terms of the b_r. We solve these equations from the bottom up; e.g. for quadratic h

$$\begin{aligned} c_2 &= \frac{b_2}{a_0}, \\ c_1 &= \frac{b_1 - 2a_1c_2}{a_0}, \\ c_0 &= \frac{b_0 - a_1c_1 - 2a_2c_2}{a_0}. \end{aligned} \tag{2.27}$$

Notice that the procedure doesn't work if $a_0 = 0$; the orders of the polynomials on left and right then inevitably disagree. This difficulty may be resolved by recognizing that the equation is then a first-order one for $g \equiv f'$ and using a trial solution for g that contains a term in x^2.

Example 2.7

Find the PI for

$$f'' + 2f' + f = 1 + 2x + 3x^2.$$

Try $f = c_0 + c_1x + c_2x^2$; have

$$\left.\begin{array}{lr} x^2: & c_2 = 3 \\ x^1: & 4c_2 + c_1 = 2 \\ x^0: & 2c_2 + 2c_1 + c_0 = 1 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{l} c_1 = 2(1 - 2c_2) = -10 \\ c_0 = 1 - 2(c_2 + c_1) = 1 - 2(3 - 10) = 15 \end{array}$$

Check

$$\begin{aligned} f &= 15 - 10x + 3x^2, \\ 2f' &= (-10 + 6x) \times 2, \\ f'' &= 6, \\ L(f) &= 1 + 2x + 3x^2. \end{aligned}$$

2.6.2 Exponential f When $h = He^{\gamma x}$, we try $f = Pe^{\gamma x}$. Substituting this into the general second-order equation with constant coefficients we obtain

$$P(a_2\gamma^2 + a_1\gamma + a_0)e^{\gamma x} = He^{\gamma x}. \tag{2.28}$$

Cancelling the exponentials, solving for P, and substituting the result into $f = Pe^{\gamma x}$, we have finally

$$\begin{aligned} f &= \frac{He^{\gamma x}}{a_2\gamma^2 + a_1\gamma + a_0} \\ &= \frac{He^{\gamma x}}{a_2(\gamma - \alpha_-)(\gamma - \alpha_+)} \quad \text{where} \quad \text{CF} = A_\pm e^{\alpha_\pm x}. \end{aligned} \tag{2.29}$$

Example 2.8

Find the PI for

$$f'' + 3f' + 2f = e^{2x}.$$

So the PI is $f = \dfrac{e^{2x}}{4 + 6 + 2} = \frac{1}{12}e^{2x}$.

If h contains two or more exponentials, we find separate PIs for each of them, and then add our results to get the overall PI.

Example 2.9

Find the PI for

$$f'' + 3f' + 2f = \mathrm{e}^{2x} + 2\mathrm{e}^{x}.$$

Reasoning as above we conclude that $f_1 \equiv \frac{1}{12}\mathrm{e}^{2x}$ satisfies $f_1'' + 3f_1' + 2f_1 = \mathrm{e}^{2x}$.

and $f_2 \equiv \dfrac{2\mathrm{e}^x}{1+3+2} = \frac{1}{3}\mathrm{e}^x$ satisfies $f_2'' + 3f_2' + 2f_2 = \mathrm{e}^x$,

so $\frac{1}{12}\mathrm{e}^{2x} + \frac{1}{3}\mathrm{e}^x$ satisfies the given equation.

From equation (2.29) it is clear that we have problem when part of h is in the CF because then one of the denominators of our PI vanishes. The problem we have to address is the solution of

$$Lf = a_2\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_1\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_2\Big)f = H\mathrm{e}^{\alpha_2 x}. \tag{2.30}$$

$P\mathrm{e}^{\alpha_2 x}$ is not a useful trial function for the PI because $L\mathrm{e}^{\alpha_2 x} = 0$. Instead we try $Px\mathrm{e}^{\alpha_2 x}$. We have

$$\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_2\Big)Px\mathrm{e}^{\alpha_2 x} = P\mathrm{e}^{\alpha_2 x}, \tag{2.31}$$

and

$$L\big(Px\mathrm{e}^{\alpha_2 x}\big) = a_2\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha_1\Big)P\mathrm{e}^{\alpha_2 x} = a_2 P(\alpha_2 - \alpha_1)\mathrm{e}^{\alpha_2 x}. \tag{2.32}$$

Hence, we can solve for P so long as $\alpha_2 \neq \alpha_1$: $P = \dfrac{H}{a_2(\alpha_2 - \alpha_1)}$.

Example 2.10

Find the PI for

$$f'' + 3f' + 2f = \mathrm{e}^{-x}.$$

The CF is $A\mathrm{e}^{-2x} + B\mathrm{e}^{-x}$, so we try $f = Px\mathrm{e}^{-x}$. We require

$$\begin{aligned}\mathrm{e}^{-x} &= \Big(\frac{\mathrm{d}}{\mathrm{d}x} + 2\Big)\Big(\frac{\mathrm{d}}{\mathrm{d}x} + 1\Big)Px\mathrm{e}^{-x} = \Big(\frac{\mathrm{d}}{\mathrm{d}x} + 2\Big)P\mathrm{e}^{-x}\\ &= P\mathrm{e}^{-x}.\end{aligned}$$

Thus $P = 1$ and $f = x\mathrm{e}^{-x}$.

What if $\alpha_1 = \alpha_2 = \alpha$ and $h = H\mathrm{e}^{\alpha x}$? Then we try $f = Px^2\mathrm{e}^{\alpha x}$:

$$\begin{aligned}H\mathrm{e}^{\alpha x} &= a_2\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\Big)^2 Px^2\mathrm{e}^{\alpha x} = a_2\Big(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\Big)2Px\mathrm{e}^{\alpha x}\\ &= 2a_2 P\mathrm{e}^{\alpha x} \quad \Rightarrow \quad P = \frac{H}{2a_2}\end{aligned}$$

2.6.3 Sinusoidal h

Suppose $h = H\cos x$, so $Lf \equiv a_2 f'' + a_1 f' + a_0 f = H\cos x$.

Clumsy method:

$$f = A\cos x + B\sin x$$

$$\cdots\cdots$$

Elegant method: Find solulutions $z(x)$ of the *complex* equation

$$Lz = H\mathrm{e}^{\mathrm{i}x}. \tag{2.33}$$

Since L is real

$$\Re\mathrm{e}(Lz) = L[\Re\mathrm{e}(z)] = \Re\mathrm{e}(H\mathrm{e}^{\mathrm{i}x}) = H\Re\mathrm{e}(\mathrm{e}^{\mathrm{i}x}) = H\cos x, \tag{2.34}$$

so the real part of our solution z will answer the given problem.

Set $z = P\mathrm{e}^{\mathrm{i}x}$ (P complex)

$$Lz = (-a_2 + \mathrm{i}a_1 + a_0)P\mathrm{e}^{\mathrm{i}x} \quad \Rightarrow \quad P = \frac{H}{-a_2 + \mathrm{i}a_1 + a_0}. \tag{2.35}$$

Finally,

$$\begin{aligned} f &= H\Re\mathrm{e}\Big(\frac{\mathrm{e}^{\mathrm{i}x}}{(a_0 - a_2) + \mathrm{i}a_1}\Big) \\ &= H\frac{(a_0 - a_2)\cos x + a_1 \sin x}{(a_0 - a_2)^2 + a_1^2}. \end{aligned} \tag{2.36}$$

Note:

We shall see below that in many physical problems explicit extraction of the real part is unhelpful; more physical insight can be obtained from the first than the second of equations (2.36). But don't forget that $\Re\mathrm{e}$ operator! It's especially important to include it when evaluating the arbitrary constants in the CF by imposing initial conditions.

Example 2.11

Find the PI for

$$f'' + 3f' + 2f = \cos x.$$

We actually solve

$$z'' + 3z' + 2z = \mathrm{e}^{\mathrm{i}x}.$$

Hence

$$z = P\mathrm{e}^{\mathrm{i}x} \quad \text{where} \quad P = \frac{1}{-1 + 3\mathrm{i} + 2}.$$

Extracting the real part we have finally

$$f = \Re\mathrm{e}\Big(\frac{\mathrm{e}^{\mathrm{i}x}}{1 + 3\mathrm{i}}\Big) = \tfrac{1}{10}(\cos x + 3\sin x).$$

What do we do if $h = H\sin x$? We solve $Lz = H\mathrm{e}^{\mathrm{i}x}$ and take imaginary parts of both sides.

Example 2.12

Find the PI for

$$f'' + 3f' + 2f = \sin x.$$

Solving $z'' + 3z' + 2z = \mathrm{e}^{\mathrm{i}x}$ with $z = P\mathrm{e}^{\mathrm{i}x}$ we have

$$P = \frac{1}{1 + 3\mathrm{i}} \qquad \Rightarrow \qquad f = \Im\mathrm{m}\Big(\frac{\mathrm{e}^{\mathrm{i}x}}{1 + 3\mathrm{i}}\Big) = \tfrac{1}{10}(\sin x - 3\cos x).$$

Note:

It is often useful to express $A\cos\theta + B\sin\theta$ as $\widetilde{A}\cos(\theta + \phi)$. We do this by noting that $\cos(\theta + \phi) = \cos\phi\cos\theta - \sin\phi\sin\theta$, so

$$\begin{aligned} A\cos\theta + B\sin\theta &= \sqrt{A^2 + B^2}\Big(\frac{A}{\sqrt{A^2 + B^2}}\cos\theta + \frac{B}{\sqrt{A^2 + B^2}}\sin\theta\Big) \\ &= \sqrt{A^2 + B^2}\cos(\theta + \phi), \end{aligned}$$

where $\cos\phi = A/\sqrt{A^2 + B^2}$ and $\sin\phi = -B/\sqrt{A^2 + B^2}$.

Example 2.13
Find the PI for

$$f'' + 3f' + 2f = 3\cos x + 4\sin x.$$

The right-hand side can be rewritten $5\cos(x+\phi) = 5\Re\mathrm{e}(\mathrm{e}^{\mathrm{i}(x+\phi)})$, where $\phi = \arctan(-4/3)$. So our trial solution of the underlying complex equation is $z = P\mathrm{e}^{\mathrm{i}(x+\phi)}$. Plugging this into the equation, we find

$$P = \frac{5}{-1 + 3\mathrm{i} + 2} = \frac{5}{1 + 3\mathrm{i}},$$

so the required PI is

$$f_0 = 5\Re\mathrm{e}\left(\frac{\mathrm{e}^{\mathrm{i}(x+\phi)}}{1 + 3\mathrm{i}}\right) = \tfrac{1}{2}\left[\cos(x+\phi) + 3\sin(x+\phi)\right].$$

The last three examples are rather easy because $\mathrm{e}^{\mathrm{i}x}$ does not occur in the CF (which is $A\mathrm{e}^{-x} + B\mathrm{e}^{-2x}$). What if $\mathrm{e}^{\mathrm{i}x}$ is in the CF? Then we try $z = Px\mathrm{e}^{\mathrm{i}x}$.

Example 2.14
Find the PI for

$$f'' + f = \cos x \quad \Rightarrow \quad z'' + z = \mathrm{e}^{\mathrm{i}x}$$

From the auxilliary equation we find that the equation can be written

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\right) z = \mathrm{e}^{\mathrm{i}x}.$$

For the PI $Px\mathrm{e}^{\mathrm{i}x}$ we require

$$\mathrm{e}^{\mathrm{i}x} = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\right) Px\mathrm{e}^{\mathrm{i}x} = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}\right) P\mathrm{e}^{\mathrm{i}x} = 2\mathrm{i}P\mathrm{e}^{\mathrm{i}x}$$

$$\Rightarrow \quad P = \frac{1}{2\mathrm{i}} \quad \Rightarrow \quad f = \Re\mathrm{e}\left(\frac{x\mathrm{e}^{\mathrm{i}x}}{2\mathrm{i}}\right) = \tfrac{1}{2}x\sin x$$

2.6.4 Exponentially decaying sinusoidal h Since we are handling sinusoids by expressing them in terms of exponentials, essentially nothing changes if we are confronted by a combination of an exponential and sinusoids:

Example 2.15
Find the PI for

$$f'' + f = \mathrm{e}^{-x}\left(3\cos x + 4\sin x\right).$$

The right-hand side can be rewritten $5\mathrm{e}^{-x}\cos(x+\phi) = 5\Re\mathrm{e}(\mathrm{e}^{(\mathrm{i}-1)x+\mathrm{i}\phi})$, where $\phi = \arctan(-4/3)$. So our trial solution of the underlying complex equation is $z = P\mathrm{e}^{(\mathrm{i}-1)x+\mathrm{i}\phi}$. Plugging this into the equation, we find

$$P = \frac{5}{(\mathrm{i}-1)^2 + 1} = \frac{5}{1 - 2\mathrm{i}}.$$

Finally the required PI is

$$f_0 = 5\Re\mathrm{e}\left(\frac{\mathrm{e}^{(\mathrm{i}-1)x+\mathrm{i}\phi}}{1 - 2\mathrm{i}}\right) = \mathrm{e}^{-x}\left[\cos(x+\phi) - 2\sin(x+\phi)\right].$$

3 Application to Oscillators

Second-order differential equations with constant coefficients arise from all sorts of physical systems in which something undergoes small oscillations about a point of equilibrium. It is hard to exaggerate the importance for physics of such systems. Obvious examples include the escapement spring of a watch, the horn of a loudspeaker and an irritating bit of trim that makes a noise at certain speeds in the car. Less familiar examples include the various fields that the vacuum sports, which include the electromagnetic field and the fields whose excitations we call electrons and quarks.

The equation of motion of a mass that oscillates in response to a periodic driving force $mF\cos\omega t$ is

$$m\ddot{x} = -\underbrace{m\omega_0^2 x}_{\text{spring}} - \underbrace{m\gamma\dot{x}}_{\text{friction}} + \underbrace{mF\cos\omega t}_{\text{forcing}}. \tag{3.1}$$

Gathering the homogeneous and inhomogeneous terms onto the left- and right-hand sides, respectively, we see that the associated complex equation is

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = F\mathrm{e}^{\mathrm{i}\omega t}. \tag{3.2}$$

3.1 Transients

The auxilliary equation of (3.2) is

$$\begin{aligned} \alpha^2 + \gamma\alpha + \omega_0^2 = 0 \quad \Rightarrow \quad \alpha &= -\tfrac{1}{2}\gamma \pm \mathrm{i}\sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2} \\ &= -\tfrac{1}{2}\gamma \pm \mathrm{i}\omega_\gamma \quad \text{where} \quad \omega_\gamma \equiv \omega_0\sqrt{1 - \tfrac{1}{4}\gamma^2/\omega_0^2}. \end{aligned}$$

Hence the CF is

$$x = \mathrm{e}^{-\gamma t/2}\big[A\cos(\omega_\gamma t) + B\sin(\omega_\gamma t)\big] = \mathrm{e}^{-\gamma t/2}\widetilde{A}\cos(\omega_\gamma t + \psi), \tag{3.3}$$

where ψ, the **phase angle**, is an arbitrary constant. Since $\gamma > 0$, we have that the CF $\to 0$ as $t \to \infty$. Consequently, the part of motion that is decsribed by the CF is called the **transient** response.

3.2 Steady-state solutions

The PI of equation (3.2) is

$$x = \Re\mathrm{e}\Big(\frac{F\mathrm{e}^{\mathrm{i}\omega t}}{\omega_0^2 - \omega^2 + \mathrm{i}\omega\gamma}\Big). \tag{3.4}$$

The PI describes the steady-state response that remains after the transient has died away.

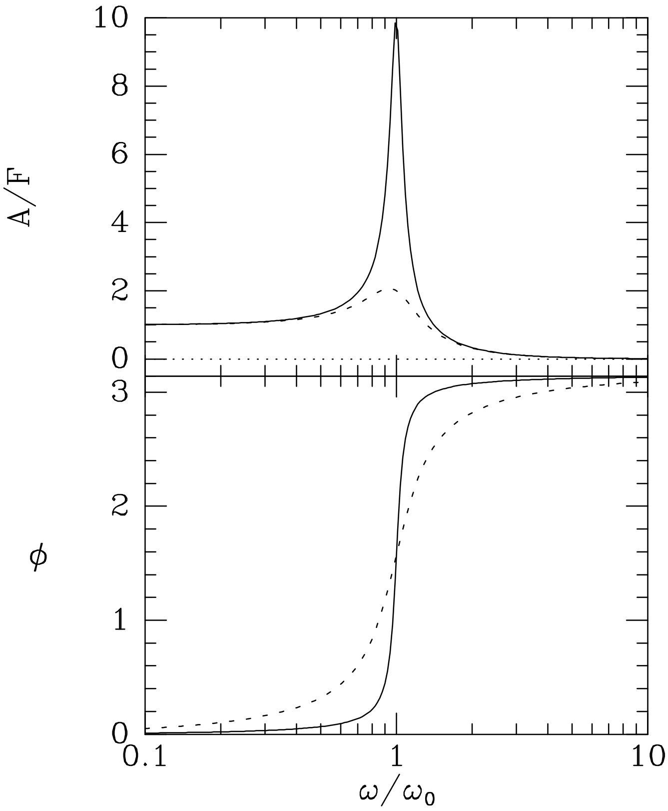
In (3.4) the bottom $= \sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\,\mathrm{e}^{\mathrm{i}\phi}$, where $\phi \equiv \arctan\Big(\dfrac{\omega\gamma}{\omega_0^2 - \omega^2}\Big)$, so the PI may also be written

$$x = \frac{F\Re\mathrm{e}\big(\mathrm{e}^{\mathrm{i}(\omega t - \phi)}\big)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} = \frac{F\cos(\omega t - \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}. \tag{3.5}$$

For $\phi > 0$, x achieves the same phase as F at t greater by ϕ/ω, so ϕ is called the **phase lag** of the response.

The amplitude of the response is

$$A = \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}, \tag{3.6}$$



Amplitude and phase of a driven oscillator. Full lines are for $\gamma = 0.1\omega_0$, dashed lines for $\gamma = 0.5\omega_0$.

which peaks when

$$0 = \frac{\mathrm{d}A^{-2}}{\mathrm{d}\omega} \propto -4(\omega_0^2 - \omega^2)\omega + 2\omega\gamma^2 \quad \Rightarrow \quad \omega^2 = \omega_0^2 - \tfrac{1}{2}\gamma^2. \tag{3.7}$$

$\omega_R \equiv \sqrt{\omega_0^2 - \gamma^2/2}$ is called the **resonant** frequency. Note that the frictional coefficient γ causes ω_R to be smaller than the natural frequency ω_0 of the undamped oscillator.

The figure shows that the amplitude of the steady-state response becomes very large at $\omega = \omega_R$ if γ/ω_0 is small. The figure also shows that the phase lag of the response increases from small values at $\omega < \omega_R$ to π at high frequencies. Many important physical processes, including dispersion of light in glass, depend on this often rapid change in phase with frequency.

3.2.1 Power input Power in is $W = \mathcal{F}\dot{x}$, where $\mathcal{F} = mF\cos\omega t$. Since $\Re\mathrm{e}(z_1) \times \Re\mathrm{e}(z_2) \neq \Re\mathrm{e}(z_1 z_2)$, we have to extract real bits before multiplying them together

$$\begin{aligned} W = \mathcal{F}\dot{x} &= \Re\mathrm{e}(mF\mathrm{e}^{\mathrm{i}\omega t}) \times \frac{\Re\mathrm{e}\big(i\omega F\mathrm{e}^{\mathrm{i}(\omega t-\phi)}\big)}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} \\ &= \frac{\omega m F^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}[-\cos(\omega t)\sin(\omega t-\phi)] \\ &= -\frac{\frac{1}{2}\omega m F^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}[\sin(2\omega t-\phi)+\sin(-\phi)]. \end{aligned} \tag{3.8}$$

Averaging over an integral number of periods, the mean power is

$$\overline{W} = \frac{\frac{1}{2}\omega m F^2 \sin\phi}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}. \tag{3.9}$$

3.2.2 Energy dissipated Let's check that the mean power input is equal to the rate of dissipation of energy by friction. The dissipation rate is

$$\overline{D} = m\gamma\overline{\dot{x}\dot{x}} = \frac{m\gamma\omega^2 F^2 \frac{1}{2}}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}. \tag{3.10}$$

It is equal to work done because $\sin\phi = \gamma\omega/\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$.

3.2.3 Quality factor Now consider the energy content of the transient motion that the CF describes. By (3.3) its energy is

$$\begin{aligned} E &= \tfrac{1}{2}(m\dot{x}^2 + m\omega_0^2 x^2) \\ &= \tfrac{1}{2}mA^2 e^{-\gamma t}\left[\tfrac{1}{4}\gamma^2\cos^2\eta + \omega_\gamma\gamma\cos\eta\sin\eta + \omega_\gamma^2\sin^2\eta + \omega_0^2\cos^2\eta\right] \quad (\eta \equiv \omega_\gamma t + \psi) \end{aligned} \tag{3.11}$$

For small γ/ω_0 this becomes

$$E \simeq \tfrac{1}{2}m(\omega_0 A)^2 e^{-\gamma t}. \tag{3.12}$$

We define the **quality factor** Q to be

$$\begin{aligned} Q &\equiv \frac{E(t)}{E(t - \pi/\omega_0) - E(t + \pi/\omega_0)} \simeq \frac{1}{e^{\pi\gamma/\omega_0} - e^{-\pi\gamma/\omega_0}} = \tfrac{1}{2}\operatorname{csch}(\pi\gamma/\omega_0) \\ &\simeq \frac{\omega_0}{2\pi\gamma} \quad \text{(for small } \gamma/\omega_0\text{).} \end{aligned} \tag{3.13}$$

Q is the inverse of the fraction of the oscillator's energy that is dissipated in one period. It is approximately equal to the number of oscillations conducted before the energy decays by a factor of e.

4 Systems of Linear DE's with Constant Coefficients

Many physical systems require more than one variable to quantify their configuration: for example a circuit might have two connected current loops, so one needs to know what current is flowing in each loop at each moment. A set of differential equations – one for each variable – will determine the dynamics of such a system. If these equations are linear and have constant coefficients, the procedure for solving them is a minor extension of the procedure for solving a single linear differential equation with constant coefficients.

The steps are:

1. Arrange the equations so that terms on the left are all proportional to an unknown variable, and already known terms are on the right.
2. Find the general solution of the equations that are obtained by setting the right sides to zero. The result of this operation is the CF. It is usually found by replacing the unknown variables by multiples of $e^{\alpha t}$ (if t is the independent variable) and solving the resulting alegraic equations.
3. Find a particular integral by putting in a trial solution for each term – polynomial, exponential, etc. – on the right hand side.

This recipe is best illustrated by some examples.

Example 4.1

Solve

$$\begin{aligned} \frac{dx}{dt} + \frac{dy}{dt} + y &= t, \\ -\frac{dy}{dt} + 3x + 7y &= e^{2t} - 1. \end{aligned}$$

It is helpful to introduce the shorthand

$$\mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} & +\frac{\mathrm{d}y}{\mathrm{d}t} + y \\ 3x & -\frac{\mathrm{d}y}{\mathrm{d}t} + 7y \end{pmatrix}.$$

To find CF

$$\text{Set} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X\mathrm{e}^{\alpha t} \\ Y\mathrm{e}^{\alpha t} \end{pmatrix} \quad \alpha, X, Y \text{ complex nos to be determined}$$

Plug into $\mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} = 0$ and cancel the factor $\mathrm{e}^{\alpha t}$

$$\begin{aligned} \alpha X + (\alpha+1)Y &= 0, \\ 3X + (7-\alpha)Y &= 0. \end{aligned} \tag{4.1}$$

The theory of equations, to be discussed early next term, shows that these equations allow X, Y to be non-zero only if the determinant

$$\begin{vmatrix} \alpha & \alpha+1 \\ 3 & 7-\alpha \end{vmatrix} = 0,$$

which in turn implies that $\alpha(7-\alpha) - 3(\alpha+1) = 0 \quad \Rightarrow \quad \alpha = 3,\ \alpha = 1$. We can arrive at the same conclusion less quickly by using the second equation to eliminate Y from the first equation. So the bottom line is that $\alpha = 3, 1$ are the only two viable values of α. For each value of α either of equations (4.1) imposes a ratio* X/Y

$\alpha = 3 \quad \Rightarrow \quad 3X + 4Y = 0 \quad \Rightarrow \quad Y = -\frac{3}{4}X,$

$\alpha = 1 \quad \Rightarrow \quad X + 2Y = 0 \quad \Rightarrow \quad Y = -\frac{1}{2}X.$

Hence the CF made up of

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} \mathrm{e}^{3t} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} = X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \mathrm{e}^{t}.$$

The two arbitrary constants in this CF reflect the fact that the original equations were first-order in two variables.

To find PI

(i) *Polynomial part*

$$\text{Try} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X_0 + X_1 t \\ Y_0 + Y_1 t \end{pmatrix}$$

Plug into $\mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -1 \end{pmatrix}$

$$\begin{array}{cc} X_1 + Y_1 + Y_1 t + Y_0 = t & 3(X_0 + X_1 t) - Y_1 + 7(Y_0 + Y_1 t) = -1 \\ \Downarrow & \Downarrow \\ Y_1 = 1;\ X_1 + Y_1 + Y_0 = 0 & 3X_0 + 7Y_0 = 0;\ 3X_1 + 7Y_1 = 0 \\ \Downarrow & \Downarrow \\ X_1 + Y_0 = -1 & X_1 = -\frac{7}{3} \end{array}$$

* The allowed values of α are precisely those for which you get the same value of X/Y from both of equations (4.1).

Consequently, $Y_0 = -1 + \frac{7}{3} = \frac{4}{3}$ and $X_0 = -\frac{7}{3}Y_0 = -\frac{28}{9}$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{pmatrix}$$

(ii) *Exponential part*

$$\text{Try} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \mathrm{e}^{2t}$$

Plug into $\mathbf{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{e}^{2t} \end{pmatrix}$ and find

$$\begin{aligned} 2X + (2+1)Y = 0 \quad &\Rightarrow \quad X = -\tfrac{3}{2}Y \\ 3X + (-2+7)Y = 1 \quad &\Rightarrow \quad (-\tfrac{9}{2} + 5)Y = 1 \end{aligned}$$

Hence $Y = 2$, $X = -3$.

Putting everything together the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} \mathrm{e}^{3t} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \mathrm{e}^{t} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} \mathrm{e}^{2t} + \begin{pmatrix} -\frac{28}{9} - \frac{7}{3}t \\ \frac{4}{3} + t \end{pmatrix} \tag{4.2}$$

We can use the arbitrary constants in the above solution to obtain a solution in which x and y or $\dot{x}$ and $\dot{y}$ take on any prescribed values at $t = 0$:

Example 4.2

For the differential equations of Example 4.1, find the solution in which

$$\begin{aligned} \dot{x}(0) &= -\tfrac{19}{3} \\ \dot{y}(0) &= 3 \end{aligned}$$

Solution: Evaluate the time derivative of the GS at $t = 0$ and set the result equal to the given data:

$$\begin{pmatrix} -\frac{19}{3} \\ 3 \end{pmatrix} = 3X_a \begin{pmatrix} 1 \\ -\frac{3}{4} \end{pmatrix} + X_b \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + 2\begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{7}{3} \\ 1 \end{pmatrix}$$

Hence

$$\begin{aligned} 3X_a + X_b &= 2 \\ -\tfrac{9}{4}X_a - \tfrac{1}{2}X_b &= -2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X_a &= \frac{-2}{-3/2} = \tfrac{4}{3} \\ X_b &= 2 - 3X_a = -2 \end{aligned}$$

Here's another, more complicated example.

Example 4.3

Solve

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= 2\sin t + 3\cos t + 5\mathrm{e}^{-t} \\ \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - y &= 3\cos t - 5\sin t - \mathrm{e}^{-t} \end{aligned} \quad \text{given} \quad \begin{aligned} x(0) = 2; \quad y(0) = -3 \\ \dot{x}(0) = 0; \quad \dot{y}(0) = 4 \end{aligned}$$

To find CF

Set $x = X\mathrm{e}^{\alpha t}$, $y = Y\mathrm{e}^{\alpha t}$

$$\Rightarrow \quad \begin{matrix} (\alpha^2+2)X & + & \alpha Y \\ \alpha X & + & (\alpha^2 - 1)Y \end{matrix} = 0 \quad \Rightarrow \quad \alpha^4 = 2$$

$$\Rightarrow \quad \alpha^2 = \pm\sqrt{2} \quad \Rightarrow \quad \alpha = \pm\beta,\ \pm\mathrm{i}\beta \quad (\beta \equiv 2^{1/4})$$

and $Y/X = -(\alpha^2+2)/\alpha$ so the CF is

$$\begin{pmatrix} x \\ y \end{pmatrix} = X_a \begin{pmatrix} \beta \\ 2+\sqrt{2} \end{pmatrix} \mathrm{e}^{-\beta t} + X_b \begin{pmatrix} -\beta \\ 2+\sqrt{2} \end{pmatrix} \mathrm{e}^{\beta t} + X_c \begin{pmatrix} \mathrm{i}\beta \\ 2-\sqrt{2} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\beta t} + X_d \begin{pmatrix} -\mathrm{i}\beta \\ 2-\sqrt{2} \end{pmatrix} \mathrm{e}^{\mathrm{i}\beta t}$$

Notice that the functions multiplying X_c and X_d are complex conjugates of one another. So if the solution is to be real X_d has to be the complex conjugate of X_c and these two complex coefficients contain only two real arbitrary constants between them. There are four arbitrary constants in the CF because we are solving second-order equations in two dependent variables.

To Find PI

Set $(x,y) = (X,Y)\mathrm{e}^{-t} \quad \Rightarrow$

$$\begin{aligned} X - Y + 2X &= 5 \\ -X + Y - Y &= -1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X &= 1 \\ Y &= -2 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-t}$$

Have $2\sin t + 3\cos t = \Re\mathrm{e}(\sqrt{13}\mathrm{e}^{\mathrm{i}(t+\phi)})$, where $\cos\phi = 3/\sqrt{13}$, $\sin\phi = -2/\sqrt{13}$.
Similarly $3\cos t - 5\sin t = \Re\mathrm{e}(\sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)})$, where $\cos\psi = 3/\sqrt{34}$, $\sin\psi = 5/\sqrt{34}$
Set $(x,y) = \Re\mathrm{e}[(X,Y)\mathrm{e}^{\mathrm{i}t}]$ and require

$$\begin{aligned} -X + \mathrm{i}Y + 2X = X + \mathrm{i}Y &= \sqrt{13}\mathrm{e}^{\mathrm{i}\phi} \\ \mathrm{i}X - Y - Y = \mathrm{i}X - 2Y &= \sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \end{aligned} \quad \Rightarrow \quad \begin{aligned} -\mathrm{i}Y &= \sqrt{13}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{i}\sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \\ \mathrm{i}X &= 2\mathrm{i}\sqrt{13}\mathrm{e}^{\mathrm{i}\phi} - \sqrt{34}\mathrm{e}^{\mathrm{i}\psi} \end{aligned}$$

so

$$\begin{aligned} x &= \Re\mathrm{e}(2\sqrt{13}\mathrm{e}^{\mathrm{i}(t+\phi)} + \mathrm{i}\sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)}) \\ &= 2\sqrt{13}(\cos\phi\cos t - \sin\phi\sin t) - \sqrt{34}(\sin\psi\cos t + \cos\psi\sin t) \\ &= 2[3\cos t + 2\sin t] - 5\cos t - 3\sin t \\ &= \cos t + \sin t \end{aligned}$$

Similarly

$$\begin{aligned} y &= \Re\mathrm{e}(\sqrt{13}\mathrm{i}\mathrm{e}^{\mathrm{i}(t+\phi)} - \sqrt{34}\mathrm{e}^{\mathrm{i}(t+\psi)}) \\ &= \sqrt{13}(-\sin\phi\cos t - \cos\phi\sin t) - \sqrt{34}(\cos\psi\cos t - \sin\psi\sin t) \\ &= 2\cos t - 3\sin t - 3\cos t + 5\sin t \\ &= -\cos t + 2\sin t. \end{aligned}$$

Thus the complete PI is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t + \sin t \\ -\cos t + 2\sin t \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mathrm{e}^{-t}.$$

For the initial-value problem

$$\mathrm{PI}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad ; \quad \dot{\mathrm{PI}}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$\mathrm{CF}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ; \quad \dot{\mathrm{CF}}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So the PI satisfies the initial data and $X_a = X_b = X_c = X_d = 0$.

In general the number of arbitrary constants in the general solution should be the sum of the orders of the highest derivative in each variable. There are exceptions to this rule, however, as the following example shows. This example also illustrates another general point: that before solving the given equations, one should always try to simplify them by adding a multiple of one equation or its derivative to the other.

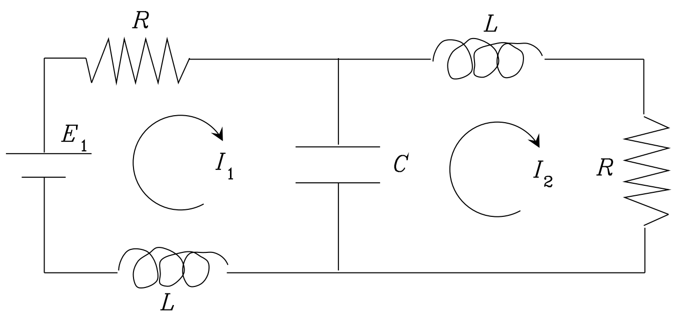
Example 4.4
Solve

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}y}{\mathrm{d}t}+y&=t,\\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+3x+7y&=\mathrm{e}^{2t}.\end{aligned}\tag{4.3}$$

We differentiate the first equation and substract the result from the seond. Then the system becomes first-order – in fact the system solved in Example 4.1. From (4.2) we see that the general solution contains only two arbitrary constants rather than the four we might have expected from a cursory glance at (4.3). To understand this phenomenon better, rewrite the equations in terms of $z\equiv x+y$ as $\dot z+z-x=t$ and $\ddot z+7z-4x=\mathrm{e}^{2t}$. The first equation can be used to make x a function $x(z,\dot z,t)$. Using this to eliminate x from the second equation we obtain an expression for $\ddot z(z,\dot z,t)$. From this expression and its derivatives w.r.t. t we can construct a Taylor series for z once we are told $z(t_0)$ and $\dot z(t_0)$. Hence the general solution should have just two arbitrary constants.

4.1 *LCR* circuits

The dynamics of a linear electrical circuit is governed by a system of linear equations with constant coefficients. These may be solved by the general technique described at the start of Chapter 4. In many cases they may be more easily solved by judicious addition and subtraction along the lines illustrated in Example 4.4.



Using Kirchhoff's laws

$$\begin{aligned}RI_1+\frac{Q}{C}+L\frac{\mathrm{d}I_1}{\mathrm{d}t}&=E_1\\ L\frac{\mathrm{d}I_2}{\mathrm{d}t}+RI_2-\frac{Q}{C}&=0.\end{aligned}\tag{4.4}$$

We first differentiate to eliminate Q

$$\begin{aligned}\frac{\mathrm{d}^2I_1}{\mathrm{d}t^2}+\frac{R}{L}\frac{\mathrm{d}I_1}{\mathrm{d}t}+\frac{1}{LC}(I_1-I_2)&=0\\ \frac{\mathrm{d}^2I_2}{\mathrm{d}t^2}+\frac{R}{L}\frac{\mathrm{d}I_2}{\mathrm{d}t}-\frac{1}{LC}(I_1-I_2)&=0.\end{aligned}\tag{4.5}$$

We now add the equations to obtain

$$\frac{\mathrm{d}^2S}{\mathrm{d}t^2}+\frac{R}{L}\frac{\mathrm{d}S}{\mathrm{d}t}=0\quad\text{where}\quad S\equiv I_1+I_2.\tag{4.6}$$

Subtracting the equations we find

$$\frac{\mathrm{d}^2D}{\mathrm{d}t^2}+\frac{R}{L}\frac{\mathrm{d}D}{\mathrm{d}t}+\frac{2}{LC}D=0\quad\text{where}\quad D\equiv I_1-I_2.\tag{4.7}$$

We now have two uncoupled equations, one for S and one for D. We solve each in the standard way (Section 2.5).

4.1.1 Time evolution of the LCR circuits The auxilliary equation for (4.6) is $\alpha^2 + R\alpha/L = 0$, and its roots are

$$\alpha = 0 \quad \Rightarrow \quad S = \text{constant} \quad \text{and} \quad \alpha = -R/L \quad \Rightarrow \quad S \propto e^{-Rt/L}. \tag{4.8}$$

Since the right side of (4.6) is zero, no PI is required.

The auxilliary equation for (4.7) is

$$\alpha^2 + \frac{R}{L}\alpha + \frac{2}{LC} = 0 \quad \Rightarrow \quad \alpha = -\tfrac{1}{2}\frac{R}{L} \pm \frac{i}{\sqrt{LC}}\sqrt{2 - \tfrac{1}{4}CR^2/L} = -\tfrac{1}{2}\frac{R}{L} \pm i\omega_R. \tag{4.9}$$

Again no PI is required.

Adding the results of (4.8) and (4.9), the general solutions to (4.6) and (4.7) are

$$I_1 + I_2 = S = S_0 + S_1 e^{-Rt/L} \quad ; \quad I_1 - I_2 = D = D_0 e^{-Rt/2L} \sin(\omega_R t + \phi).$$

From the original equations (4.5) it is easy to see that the steady-state currents are $I_1 = I_2 = \frac{1}{2}S_0 = \frac{1}{2}E_1/R$. Hence, the final general solution is

$$\begin{aligned} I_1 + I_2 &= S(t) = K e^{-Rt/L} + \frac{E_1}{R} \\ I_1 - I_2 &= D(t) = D_0 e^{-Rt/2L} \sin(\omega_R t + \phi). \end{aligned} \tag{4.10}$$

Example 4.5

The battery is first connected up at $t = 0$. Determine I_1, I_2 for $t > 0$.

Solution: We have $I_1(0) = I_2(0) = 0$ and from the diagram we see that $\dot{I}_1(0) = E_1/L$ and $\dot{I}_2 = 0$. Looking at equations (4.10) we set $K = -E_1/R$ to ensure that $I_1(0) + I_2(0) = 0$, and $\phi = 0$ to ensure that $I_1(0) = I_2(0)$. Finally we set $D_0 = \dfrac{E_1}{L\omega_R}$ to ensure that $\dot{D}(0) = \dfrac{E_1}{L}$

5 Non-Linear Equations

Non-linear equations are generally not soluble analytically – in large measure because their solutions display richer structure than analytic functions can describe. There are some interesting special cases, however, in which analytic solutions can be derived.

5.1 Homogenoeous equations

Consider equations of the form

$$\frac{dy}{dx} = f(y/x). \tag{5.1}$$

Such equations are called **homogeneous** because they are invariant under a rescaling of both variables: that is, if $X = sx$, $Y = sy$ are rescaled variables, the equation for $Y(X)$ is identical to that for $y(x)$. These equations are readily solved by the substitution

$$y = vx \quad \Rightarrow \quad y' = v'x + v. \tag{5.2}$$

We find

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} = \ln x + \text{constant}. \tag{5.3}$$

Example 5.1

Solve

$$xy\frac{dy}{dx} - y^2 = (x + y)^2 e^{-y/x}.$$

Solution: Dividing through by xy and setting $y = vx$ have

$$(v'x + v) - v = \frac{(1+v)^2}{v}\mathrm{e}^{-v} \quad \Rightarrow \quad \ln x = \int \frac{\mathrm{e}^v v\mathrm{d}v}{(1+v)^2}.$$

The substitution $u \equiv 1 + v$ transforms integral to

$$\mathrm{e}^{-1}\int \Big(\frac{1}{u} - \frac{1}{u^2}\Big)\mathrm{e}^u \mathrm{d}u = \mathrm{e}^{-1}\Big[\frac{\mathrm{e}^u}{u}\Big].$$

5.2 Exact equations

Suppose x, y are related by $\phi(x,y) = 0$. Then $0 = \mathrm{d}\phi = \phi_x \mathrm{d}x + \phi_y \mathrm{d}y \quad (\phi_x \equiv \partial\phi/\partial x$ etc). Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\phi_x}{\phi_y} \tag{5.4}$$

Conversely, given $y' = f(x,y)$ we can ask if there exists a function $\phi(x,y)$ such that $f = \phi_x/\phi_y$.

Example 5.2

Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(3x^2 + 2xy + y^2)\tan x - (6x + 2y)}{(2x + 2y)}.$$

Solution: Notice that

$$\text{top} \times \cos x = -\frac{\partial}{\partial x}\big[(3x^2 + 2xy + y^2)\cos x\big]$$

and

$$\text{bottom} \times \cos x = \frac{\partial}{\partial y}\big[(3x^2 + 2xy + y^2)\cos x\big]$$

so the solution is $(3x^2 + 2xy + y^2)\cos x = \text{constant}$.

5.3 Equations solved by interchange of variables

Consider

$$y^2\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0.$$

As it stands the equation is non-linear, so apparently insoluble. But when we interchange the rôles of the dependent and independent variables, it becomes linear: on multiplication by $(\mathrm{d}x/\mathrm{d}y)$ get

$$y^2 + x - 2y\frac{\mathrm{d}x}{\mathrm{d}y} = 0.$$

5.4 Equations solved by linear transformation

Consider

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x - y)^2.$$

In terms of $u \equiv y - x$ the equation reads $\mathrm{d}u/\mathrm{d}x = u^2 - 1$, which is trivially soluble.

Similarly, given

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y}{x - y + 1}$$

we define

$$u \equiv x - y + 1 \quad \text{and have} \quad 1 - \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u - 1}{u} \quad \Rightarrow \quad u\frac{\mathrm{d}u}{\mathrm{d}x} = 1,$$

which is trivially soluble.

6 Green's Functions*

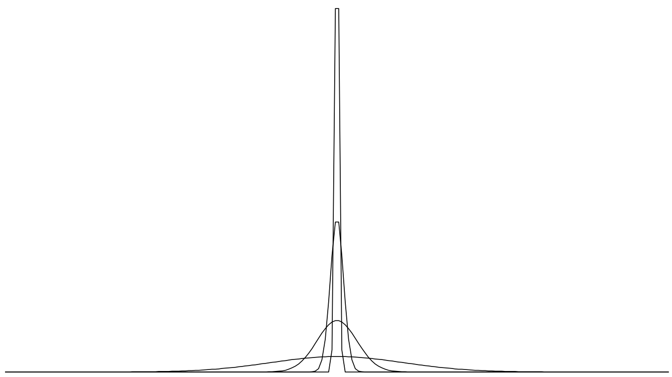
In this section we describe a powerful technique for generating particular integrals. We illustrate it by considering the general second-order linear equation

$$L_x(y) \equiv a_2(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = h(x). \qquad (6.1)$$

On dividing through by a_2 one sees that without loss of generality we can set $a_2 = 1$.

6.1 The Dirac δ-function

Consider series of ever bumpier functions such that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$, e.g.



Define $\delta(x)$ as limit of such functions. ($\delta(x)$ itself isn't a function really.) Then

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$$

δ's really important property is that

$$\int_a^b f(x)\delta(x - x_0)\,\mathrm{d}x = f(x_0) \quad \forall \begin{cases} a < x_0 < b \\ f(x) \end{cases}$$

Exercises (1):

(i) Prove that $\delta(ax) = \delta(x)/|a|$. If x has units of length, what dimensions has δ?

(ii) Prove that $\delta(f(x)) = \sum_{x_k} \delta(x - x_k)/|f'(x_k)|$, where the x_k are all points satisfying $f(x_k) = 0$.

6.2 Defining the Green's function

Now suppose for each fixed x_0 we had the function $G_{x_0}(x)$ such that

$$L_x G_{x_0} = \delta(x - x_0). \qquad (6.2)$$

Then we could easily obtain the desired PI:

$$y(x) \equiv \int_{-\infty}^{\infty} G_{x_0}(x)h(x_0)\,\mathrm{d}x_0. \qquad (6.3)$$

y is the PI because

$$\begin{aligned} L_x(y) &= \int_{-\infty}^{\infty} L_x G_{x_0}(x)h(x_0)\,\mathrm{d}x_0 \\ &= \int \delta(x - x_0)h(x_0)\,\mathrm{d}x_0 \\ &= h(x). \end{aligned}$$

Hence, once you have the **Green's function** G_{x_0} you can easily find solutions for various h.

* Lies beyond the syllabus

6.3 Finding G_{x_0}

Let $y = v_1(x)$ and $y = v_2(x)$ be two linearly independent solutions of $L_x y = 0$ – i.e. let the CF of our equation be $y = Av_1(x) + Bv_2(x)$. At $x \neq x_0$, $L_x G_{x_0} = 0$, so $G_{x_0} = A(x_0)v_1(x) + B(x_0)v_2(x)$. But in general we will have different expressions for G_{x_0} in terms of the v_i for $x < x_0$ and $x > x_0$:

$$G_{x_0} = \begin{cases} A_-(x_0)v_1(x) + B_-(x_0)v_2(x) & x < x_0 \\ A_+(x_0)v_1(x) + B_+(x_0)v_2(x) & x > x_0. \end{cases} \tag{6.4}$$

We need to choose the four functions $A_\pm(x_0)$ and $B_\pm(x_0)$. We do this by:

(i) obliging G_{x_0} to satisfy boundary conditions at $x = x_{\min} < x_0$ and $x = x_{\max} > x_0$ (e.g. $\lim_{x\to\pm\infty} G_{x_0} = 0$);

(ii) ensuring $L_x G_{x_0} = \delta(x - x_0)$.

We deal with (i) by defining $u_\pm \equiv P_\pm v_1 + Q_\pm v_2$ with $P_\pm, Q_\pm$ chosen s.t. u_- satisfies given boundary condition at $x = x_{\min}$ and u_+ satisfies condition at $x_{\max}$. Then

$$G_{x_0}(x) = \begin{cases} C_-(x_0)u_-(x) & x < x_0, \\ C_+(x_0)u_+(x) & x > x_0. \end{cases} \tag{6.5}$$

We get $C_\pm$ by integrating the differential equation from $x_0 - \epsilon$ to $x_0 + \epsilon$:

$$\begin{aligned} 1 &= \int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x - x_0)\,\mathrm{d}x = \int_{x_0-\epsilon}^{x_0+\epsilon} L_x G_{x_0}\,\mathrm{d}x \\ &= \int_{x_0-\epsilon}^{x_0+\epsilon} \left(\frac{\mathrm{d}^2 G_{x_0}}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}G_{x_0}}{\mathrm{d}x} + a_0(x)G_{x_0}(x) \right) \mathrm{d}x \\ &= \left[\frac{\mathrm{d}G_{x_0}}{\mathrm{d}x} + a_1(x_0)G_{x_0}(x) \right]_{x_0-\epsilon}^{x_0+\epsilon} + \int_{x_0-\epsilon}^{x_0+\epsilon} \left(a_0 - \frac{\mathrm{d}a_1}{\mathrm{d}x} \right) G_{x_0}(x)\,\mathrm{d}x. \end{aligned} \tag{6.6}$$

We assume that $G_{x_0}(x)$ is finite and continuous at x_0, so the second term in $[\ldots]$ vanishes and the remaining integral vanishes as $\epsilon \to 0$. Then we have two equations for $C_\pm$:

$$\begin{aligned} 1 &= C_+(x_0)\left.\frac{\mathrm{d}u_+}{\mathrm{d}x}\right|_{x_0} - C_-(x_0)\left.\frac{\mathrm{d}u_-}{\mathrm{d}x}\right|_{x_0} \\ 0 &= C_+(x_0)u_+(x_0) - C_-(x_0)u_-(x_0). \end{aligned} \tag{6.7}$$

Solving for $C_\pm$ we obtain

$$C_\pm(x_0) = \left.\frac{u_\mp}{\Delta}\right|_{x_0} \quad \text{where} \quad \Delta(x_0) \equiv \left(\frac{\mathrm{d}u_+}{\mathrm{d}x}u_- - u_+\frac{\mathrm{d}u_-}{\mathrm{d}x} \right)_{x_0}. \tag{6.8}$$

Substing these solutions back into (6.5) we have finally

$$G_{x_0}(x) = \begin{cases} \dfrac{u_+(x_0)u_-(x)}{\Delta(x_0)} & x < x_0 \\ \dfrac{u_-(x_0)u_+(x)}{\Delta(x_0)} & x > x_0. \end{cases} \tag{6.9}$$

Example 6.1

Solve

$$L_x = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - k^2 y = h(x) \quad \text{subject to} \quad \lim_{x\to\pm\infty} y = 0.$$

The required complementary functions are $u_- = \mathrm{e}^{kx}$, $u_+ = \mathrm{e}^{-kx}$, so

$$\Delta(x_0) = -k\mathrm{e}^{-kx}\mathrm{e}^{kx} - \mathrm{e}^{-kx}k\mathrm{e}^{kx} = -2k.$$

Hence

$$\begin{aligned} G_{x_0}(x) &= -\frac{1}{2k}\begin{cases} \mathrm{e}^{-k(x_0-x)} & x < x_0 \\ \mathrm{e}^{k(x_0-x)} & x > x_0 \end{cases} \\ &= -\frac{1}{2k}\mathrm{e}^{-k|x_0-x|} \end{aligned}$$

and

$$y(x) = -\frac{1}{2k}\left[\mathrm{e}^{-kx}\int_{-\infty}^{x} \mathrm{e}^{kx_0}h(x_0)\,\mathrm{d}x_0 + \mathrm{e}^{kx}\int_{x}^{\infty} \mathrm{e}^{-kx_0}h(x_0)\right]\mathrm{d}x_0.$$

Suppose $h(x) = \cos x = \Re\mathrm{e}(\mathrm{e}^{\mathrm{i}x})$. Then

$$-2ky(x) = \Re\mathrm{e}\left(\mathrm{e}^{-kx}\left[\frac{\mathrm{e}^{x_0(i+k)}}{i+k}\right]_{-\infty}^{x} + \mathrm{e}^{kx}\left[\frac{\mathrm{e}^{x_0(i-k)}}{i-k}\right]_{x}^{\infty}\right)$$

So

$$y = -\frac{\cos x}{1+k^2}$$

as expected.