Classical Fields III: Solutions

1. Let $e_1 = e^{i\phi_1}$, $e_2 = e^{i\phi_2}$ with ϕ_1, ϕ_2 real, then

$$\phi = u + iv = \psi^1 e_1 + \psi^2 e_2 = (\psi^1 \cos\phi_1 + \psi^2 \cos\phi_2) + i(\psi^1 \sin\phi_1 + \psi^2 \sin\phi_2)$$

So we require

$$\psi^1 = \frac{\sin\phi_2 u - \cos\phi_2 v}{\sin(\phi_2 - \phi_1)}$$

and similarly for v. Given that $\phi_1 \neq \phi_2$ the ψ^i can be be determined.

The general covariant derivative in this case is $\nabla_\mu \psi^a = \partial_\mu \psi^a + \Gamma^a_{1\mu}\psi^1 + \Gamma^a_{2\mu}\psi^2$ which coincides with $D_\mu\psi = \partial_\mu\psi + i(q/\hbar)A_\mu\psi$ if we adopt $\psi^1 = \Re e(\psi)$, $\psi^2 = \Im m(\psi)$ $\Gamma^2_{2\mu} = \Gamma^1_{1\mu} = 0$ and $\Gamma^1_{2\mu} = -\Gamma^2_{1\mu} = -qA_\mu/\hbar$.

2.

$$\begin{aligned}\nabla_\mu\nabla_\nu Z^\alpha &= \partial_\mu \nabla_\nu Z^\alpha + \Gamma^\alpha_{\mu\beta}\nabla_\nu Z^\beta - \Gamma^\beta_{\mu\nu}\nabla_\beta Z^\alpha \\ &= \partial_\mu(\partial_\nu Z^\alpha + \Gamma^\alpha_{\nu\beta}Z^\beta) + \Gamma^\alpha_{\mu\beta}(\partial_\nu Z^\beta + \Gamma^\beta_{\nu\gamma}Z^\gamma) - \Gamma^\beta_{\mu\nu}\nabla_\beta Z^\alpha\end{aligned}$$

The part of this that is antisymmetric in $\mu\nu$ is

$$R^\alpha_{\beta\mu\nu}Z^\beta = [\partial_\mu\Gamma^\alpha_{\nu\beta} - \partial_\nu\Gamma^\alpha_{\mu\beta}]Z^\beta + [\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\gamma} - \Gamma^\alpha_{\nu\beta}\Gamma^\beta_{\mu\gamma}]Z^\gamma$$

Now

$$\begin{aligned}D_\mu D_\nu\psi &= [\partial_\mu - i(q/\hbar)A_\mu][\partial_\nu - i(q/\hbar)A_\nu]\psi \\ &= \left[\partial_\mu\partial_\nu - i(q/\hbar)(A_\mu\partial_\nu + A_\nu\partial_\mu) - i(q/\hbar)\partial_\mu A_\nu - (q/\hbar)^2 A_\mu A_\nu\right]\psi\end{aligned}$$

$R_{\mu\nu}\psi$ is the part of this that's antisymmetric in $\mu\nu$

$$R_{\mu\nu}\psi = -i(q/\hbar)(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = -i(q/\hbar)F_{\mu\nu}\psi$$

Reintroducing labels 1 and 2 for real and imaginary parts, we can read off from this $R^1_{0\mu\nu} = R^2_{2\mu\nu} = 0$ and $R^1_{2\mu\nu} = -R^2_{1\mu\nu} = (q/\hbar)F_{\mu\nu}$.

3. Extremizing the "Lagrangian"

$$-c^2 D\dot t^2 + \frac{\dot r^2}{D} + r^2\dot\theta^2 + r^2\sin^2\theta\dot\phi^2$$

we find for the t equation of motion

$$\frac{d}{d\tau}(-2c^2 D\dot t) = 0 \quad\Rightarrow\quad \ddot t + \frac{D'}{D}\dot r\dot t = 0$$

so $\Gamma^t_{rt} = \frac{1}{2}D'/D$. For a radially moving photon we have

$$0 = \frac{dx^\mu}{ds}\nabla_\mu k^0 = \frac{dk^0}{ds} + \Gamma^0_{\mu\nu}k^\mu k^\nu = \frac{dk^0}{ds} + \frac{D'}{D}k^0\frac{dr}{ds} = \frac{1}{D}\frac{d}{ds}(Dk^0)$$

so

$$\omega(r) = \frac{\omega(\infty)}{D(r)} = \frac{\omega(\infty)}{1 - r_s/r}.$$

This equation shows that as r increases ω decreases to its value at ∞ – this is the gravitational redshift in action.

4. For a $\mathbf{B}$ field along the x axis

$$F^{\mu\nu} = F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix}$$

so

$$\mathbf{F}\cdot\mathbf{F} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -B^2 & 0 \\ 0 & 0 & 0 & -B^2 \end{pmatrix}$$

and

$$T_{\mu\nu} = \frac{1}{\mu_0}[\tfrac{1}{4}\,\mathrm{Tr}(\mathbf{F}\cdot\mathbf{F})\eta_{\mu\nu} - (\mathbf{F}\cdot\mathbf{F})_{\mu\nu}] = \frac{B^2}{\mu_0}\left[-\tfrac{1}{2}\,\mathrm{diag}(-1,1,1,1) + \mathrm{diag}(0,0,1,1)\right] = \frac{B^2}{2\mu_0}\,\mathrm{diag}(1,-1,1,1)$$

so there's pressure $P = B^2/2\mu_0$ in the yz directions and tension per unit area of the same magnitude along x.

5. The energy-momentum tensor is

$$T^{\mu\nu} = \mathrm{diag}(\rho c^2, -F/A, 0, 0)$$

Consider $T'_{\mu\nu}$ in the frame boosted along x, showing only the x^0, x^1 entries:

$$T'^{\mu\nu} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} \rho c^2 & 0 \\ 0 & -F/A \end{pmatrix}\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}\begin{pmatrix} \gamma\rho c^2 & \beta\gamma\rho c^2 \\ -\beta\gamma F/A & -\gamma F/A \end{pmatrix}$$

Hence

$$T'^{00} = \gamma^2\rho c^2 - (\beta\gamma)^2 F/A = \gamma^2(\rho c^2 - \beta^2 F/A)$$

and we need $F/A < \rho c^2$ if this is to remain > 0 in the limit $\beta \to 1$.

The speed of transverse waves on the rope is

$$c_s = \sqrt{\frac{\text{tension}}{\text{mass/length}}} = \sqrt{\frac{F}{\rho A}}$$

so $F/A < \rho c^2 \quad \Leftrightarrow \quad c_s < c$.

6. Extremizing the "Lagrangian" $-c^2\dot{t}^2 + \dot{z}^2 + r_0^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$ we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}(-2c^2\dot{t}) &= 0 & \frac{\mathrm{d}}{\mathrm{d}\tau}(2\dot{z}) &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}\tau}(2r_0^2\dot{\theta}) - 2r_0^2\sin\theta\cos\theta\dot{\phi}^2 &= 0 & \frac{\mathrm{d}}{\mathrm{d}\tau}(2r_0^2\sin^2\theta\dot{\phi}) &= 0 \\ \Rightarrow \quad \ddot{\theta} - \tfrac{1}{2}\sin 2\theta\dot{\phi}^2 &= 0 & \ddot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} &= 0 \end{aligned}$$

so

$$\Gamma^t_{\mu\nu} = 0 \qquad \Gamma^z_{\mu\nu} = 0 \qquad \Gamma^\theta_{\phi\phi} = -\tfrac{1}{2}\sin 2\theta \qquad \Gamma^\phi_{\theta\phi} = \cot\theta$$

Now

$$\begin{aligned} R_{\theta\theta} &= \partial_\theta\Gamma^\mu_{\mu\theta} - \partial_\mu\Gamma^\mu_{\theta\theta} + \Gamma^\lambda_{\theta\mu}\Gamma^\mu_{\theta\lambda} - \Gamma^\mu_{\lambda\mu}\Gamma^\lambda_{\theta\theta} \\ &= \partial_\theta\cot\theta + \Gamma^\lambda_{\theta\phi}\Gamma^\phi_{\theta\lambda} = -\csc^2\theta + \cot^2\theta = -1 \end{aligned}$$

and

$$\begin{aligned}
R_{\phi\phi} &= \partial_\phi \Gamma^\mu_{\mu\phi} - \partial_\mu \Gamma^\mu_{\phi\phi} + \Gamma^\lambda_{\phi\mu}\Gamma^\mu_{\phi\lambda} - \Gamma^\mu_{\lambda\mu}\Gamma^\lambda_{\phi\phi} \\
&= \partial_\phi \cot\theta - \partial_\theta(-\tfrac{1}{2}\sin 2\theta) + \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi} + \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\phi\theta} - \Gamma^\phi_{\theta\phi}\Gamma^\theta_{\phi\phi} \\
&= \cos 2\theta + \cot\theta(-\tfrac{1}{2}\sin 2\theta) = -\sin^2\theta
\end{aligned}$$

So

$$R_{\mu\nu} = \mathrm{diag}(0, 0, -1, -\sin^2\theta)$$

$$R_\mu{}^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\sin^2\theta \end{pmatrix} \begin{pmatrix} -1/c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r_0^2 & 0 \\ 0 & 0 & 0 & 1/(r_0^2\sin^2\theta) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/r_0^2 & 0 \\ 0 & 0 & 0 & -1/r_0^2 \end{pmatrix}$$

and $R = -2/r_0^2$. Finally

$$\begin{aligned}
G_{\mu\nu} &= \mathrm{diag}(0, 0, -1, -\sin^2\theta) + r_0^{-2}\,\mathrm{diag}(-c^2, 1, r_0^2, r_0^2\sin^2\theta) \\
&= \mathrm{diag}(-c^2/r_0^2, 1/r_0^2, 0, 0) = -\frac{8\pi G}{c^4}\,\mathrm{diag}(T_{00}, T_{zz}, 0, 0)
\end{aligned}$$

so $T_{zz} = -c^4/(8\pi G r_0^2)$. The tension is

$$F = r_0^2 \int_0^{\theta_m} \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi\, T_{zz} = 2\pi(1 - \cos\theta_m)\frac{r_0^2 c^4}{8\pi G r_0^2} = \frac{c^4(1 - \cos\theta_m)}{4G}$$

7. The Minkowski metric is $-\mathrm{d}u\mathrm{d}v + \mathrm{d}y^2 + \mathrm{d}z^2$.

Extremizing the "Lagrangian" $-\dot u \dot v + f^2\dot y^2 + \dot z^2$ we obtain

$$\frac{\mathrm{d}\dot u}{\mathrm{d}\tau} = 0 \quad ; \quad -\frac{\mathrm{d}\dot v}{\mathrm{d}\tau} - 2ff'\dot y^2 - 2gg'\dot z^2 = 0; \qquad \begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}(f^2\dot y) \\ &= f^2\ddot y + 2ff'\dot u\dot y \end{aligned} \quad ; \quad \begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}(g^2\dot z) \\ &= g^2\ddot z + 2gg'\dot u\dot z \end{aligned}$$

so the non-vanishing Christoffel symbols are

$$\Gamma^v_{yy} = 2ff' \qquad \Gamma^v_{zz} = 2gg' \qquad \Gamma^y_{uy} = f'/f \qquad \Gamma^z_{uz} = g'/g$$

From eqs of motion above can see that $\dot y = 0 \quad \Rightarrow \ddot y = 0$ and similarly for z, so $y =$ const, $z =$ const are solutions. Also then $\ddot u = \ddot v$ are required, so u and v are linear in τ. If x is constant $c^2\mathrm{d}\tau^2 = \mathrm{d}u\mathrm{d}v$, which is consistent with this linearity. Thus constant x, y, z defines geodesics.

$Lf' = \Theta + u\Theta'$, $Lf'' = 2\Theta' + u\Theta''$ and $Lg' = -2\Theta' - u\Theta''$ so

$$\frac{f''}{f} + \frac{g''}{g} = \frac{2\Theta' + u\Theta''}{L + u\Theta} - \frac{2\Theta' + u\Theta''}{L - u\Theta}$$

which always vanishes because the numerators vanish unless $u = 0$, and when $u = 0$ the denominators are equal so the two terms cancel.

$$\begin{aligned}
D_z &= \int_{-a}^{a} \mathrm{d}z\, g = \int_{-a}^{a} \mathrm{d}z\,[1 - \frac{u}{L}\Theta(u)] = 2a[1 - \frac{u}{L}\Theta(u)] \\
D_y &= 2a[1 + \frac{u}{L}\Theta(u)] \\
D_x &= \int_{-a}^{a} \mathrm{d}x = 2a
\end{aligned}$$

Thus

$$D_z = \begin{cases} 2a(1 - ct/L) & \text{for } 0 < ct \\ 2a & \text{otherwise} \end{cases} \quad ; \quad D_y = \begin{cases} 2a(1 + ct/L) & \text{for } 0 < ct \\ 2a & \text{otherwise} \end{cases}$$

so at $t = 0$ particles are impelled towards each other along z and away from each other along y by a disturbance that propagates along x. This is a gravitational shock wave.

8. The eqns of $\theta\phi$ motion are

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}(2a^2r^2\dot\theta) - a^2 r \sin 2\theta\dot\phi^2$$
$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}(2a^2r^2\sin^2\theta\dot\phi)$$

so $r^2\sin^2\theta\dot\phi = \text{const}$. If this const is zero and $\theta = \pi/2$, then $\mathrm{d}(r^2\dot\theta)/\mathrm{d}\tau = 0$, which is satisfied by $\dot\theta = 0$ at all τ.

The current distance is obtained by integrating $\mathrm{d}s = a(t_0)\mathrm{d}r$ from zero to the coordinate r_g of the galaxy, and we have $D = a(t_0)r_g = r_g$ because currently $a = 1$.

Since photon propagates radially, $\mathrm{d}t = a(t)\mathrm{d}r$, and $r_g = \int_{t_1}^{t_0}\mathrm{d}t/a = \int_{t_1}^{t_0}\mathrm{d}t/(t/t_0)^{2/3} = 3t_0^{2/3}(t_0^{1/3} - t_1^{1/3})$. Hence $D = 3t_0^{2/3}(t_0^{1/3} - t_1^{1/3})$.

We have $K > 0$ because the universe is closed, so the distance to the galaxy is.

$$D = a(t_0)\int_0^{r_g}\frac{\mathrm{d}r}{\sqrt{1-Kr^2}}$$
$$= \frac{a(t_0)}{\sqrt{K}}\int_0^{\psi_g}\mathrm{d}\psi \quad \text{where} \quad \sin\psi \equiv \sqrt{K}\,r$$

Hence $\sin(\sqrt{K}D/a(t_0)) = \sqrt{K}r_g$.

At t_1 let the edge of the galaxy be at angular coordinate θ_m, so $R = a(t_1)r_g\theta_m$ and

$$\theta_m = \frac{R}{a(t_1)r_g} = \frac{\sqrt{K}R}{a(t_1)\sin(\sqrt{K}D/a(t_0))} = \frac{(1+z)\sqrt{K}R}{\sin(\sqrt{K}D)}$$

because $a(t_1) = (1+z)^{-1}$.

9.

$$u^\alpha\nabla_\alpha v^\beta - v^\alpha\nabla_\alpha u^\beta = u^\alpha\partial_\alpha v^\beta - v^\alpha\partial_\alpha u^\beta + \Gamma^\beta_{\gamma\alpha}u^\alpha v^\gamma - \Gamma^\beta_{\gamma\alpha}v^\alpha u^\gamma = [u,v]^\beta$$

by the symmetry of Γ.

$$\left[\frac{\mathrm{d}x}{\mathrm{d}\tau},\frac{\mathrm{d}x}{\mathrm{d}\epsilon}\right]^\beta = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\nabla_\alpha\frac{\mathrm{d}x^\beta}{\mathrm{d}\epsilon} - \frac{\mathrm{d}x^\alpha}{\mathrm{d}\epsilon}\nabla_\alpha\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\epsilon} - \frac{\mathrm{d}}{\mathrm{d}\epsilon}\frac{\mathrm{d}x\beta}{\mathrm{d}\tau} = 0$$

In the given definition of R we put $u^\lambda = w^\lambda = \mathrm{d}x^\lambda/\mathrm{d}\tau$ and $v^\nu = \mathrm{d}x^\nu/\mathrm{d}\epsilon$ and have

$$\left(\dot x^\alpha\nabla_\alpha\frac{\mathrm{d}x^\beta}{\mathrm{d}\epsilon}\nabla_\beta - \frac{\mathrm{d}x^\beta}{\mathrm{d}\epsilon}\nabla_\beta\dot x^\alpha\nabla_\alpha\right)\dot x^\gamma = R^\gamma{}_{\lambda\mu\nu}\dot x^\lambda\dot x^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\epsilon}$$

The second term in the brackets on the left vanishes because $\mathbf{x}(\tau)$ is geodesic. Moreover,

$$\frac{\mathrm{d}x^\beta}{\mathrm{d}\epsilon}\nabla_\beta\dot x^\alpha = \dot x^\beta\nabla_\beta\frac{\mathrm{d}x^\alpha}{\mathrm{d}\epsilon} \qquad \text{because} \qquad [,] = 0$$

so we can rewrite the first term and then have the equation of geodesic deviation:

$$(\dot x^\alpha\nabla_\alpha)(\dot x^\beta\nabla_\beta)\frac{\mathrm{d}x^\gamma}{\mathrm{d}\epsilon} = R^\gamma{}_{\lambda\mu\nu}\dot x^\lambda\dot x^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\epsilon}.$$

Dropped masses have geodesic paths $x(\tau,\epsilon)$ with $\dot x^0 \simeq c$. Since $\dot x^\alpha\nabla_\alpha = \mathrm{d}/\mathrm{d}\tau$, when we multiply the equation of geodesic deviation by a small number $\delta\epsilon$ we get

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\delta x^\gamma \simeq c^2 R^\gamma{}_{00\nu}\delta x^\nu.$$

But from elementary mechanics $\ddot z = -GM/R^2$, where M is the Earth's mass and R is the particle's distance from the centre of the Earth. Thus varying z we have

$$\delta\ddot z = \frac{2GM}{R^3}\delta z$$

Comparing with the z component of the equation of geodesic deviation and setting $g = GM/R^2$ we obtain $R^z{}_{00z} = 2g/(c^2R)$.