Functions of a complex variable (S1)

Problem sheet 2

I. Multi-valued functions; branch points and branch cuts

1. (a) Find the location and order of the branch points of the function

$$w = (z-1)^{1/3}$$

and describe a branch cut. (b) Describe a Riemann surface for this function, and determine the image of each Riemann sheet in the w plane.

2. For each of the following functions

$$(a)\ \ln\left(\frac{z-1}{z+1}\right) \quad , \qquad (b)\ \frac{\ln(z+i)}{1+z^2} \quad , \qquad (c)\ \ln(z^2-1) \quad ,$$

find location and order of the branch points, and give a valid branch cut.

3. Consider the function

$$f(z) = \sqrt{z^2+1} \ .$$

(a) Give location and order of the branch points of $f(z)$.

(b) Suppose evaluating f at the point $z = 2+2i$, then let z vary along the circle passing through $2+2i$ with centre at the origin, moving counterclockwise. When a full 2π cycle is completed by returning to the point $z = 2+2i$, determine whether or not f is restored to its initial value.

(c) Classify the behaviour of $f(z)$ at the point $z = \infty$.

(d) Describe a valid branch cut for $f(z)$ and the Riemann surface.

4. (a) Show that the inverse sine function $f(z) = \arcsin z$ is given by

$$f(z) = \arcsin z = \frac{1}{i}\ \ln(iz + \sqrt{1-z^2}) \ .$$

(b) Give the location and order of the branch points of this function.

(c) Consider the branch of $f(z) = \arcsin z$ defined using the branch cuts in Fig. 1, taking the principal branch of the logarithm and the branch of the square root such that $\sqrt{1-z^2} = 1$ when $z = 0$. Determine the value of $f(z)$ and of its derivative $f'(z)$ at the point $z = 3$.

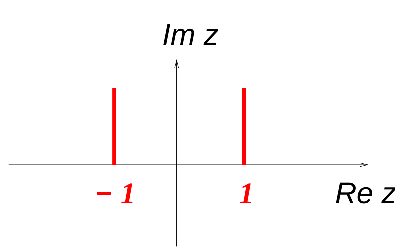

Fig. 1

5. Give the location and order of the branch points of the function

$$f(z) = \sqrt{z(z^2-1)}$$

and describe a valid branch cut.

6. Suppose that a branch of the function

$$f(z) = (z-1)^{2/3}$$

is defined by means of the branch cut in Fig. 2 and that it takes the value 1 when $z = 0$. Determine the value of $f(z)$ and of its derivative $f'(z)$ at the point $z = -i$.

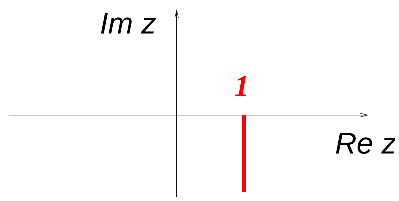

Fig. 2

II. Complex integration

7. (a) Calculate the integral

$$I = \int_L z^2 \, dz$$

where L is the straight-line segment in the complex z plane from point $z = 0$ to point $z = 2 + i$.

(b) Calculate the integrals

$$I_1 = \int_{L_1} z^2 \, dz \ , \qquad I_2 = \int_{L_2} z^2 \, dz$$

where L_1 is the straight-line segment in the complex z plane from point $z = 0$ to point $z = 2$, and L_2 is the straight-line segment from point $z = 2$ to point $z = 2 + i$.

(c) Evaluate the difference of the integrals calculated above, $I - I_1 - I_2$, and interpret the result.

8. (a) Calculate the integral

$$I = \int_\gamma \overline{z} \, dz$$

where γ is the semicircle in the upper half z plane with centre at the origin and radius 1, traveled clockwise.

(b) Calculate the integral

$$I' = \int_{\gamma'} \overline{z} \, dz$$

where γ' is the semicircle in the lower half z plane with centre at the origin and radius 1, traveled counterclockwise.

(c) Evaluate the difference of the integrals calculated above, $I' - I$, and interpret the result.

9. Let I_n be the complex integral

$$I_n = \oint_{C_{a,r}} (z-a)^n \, dz$$

where $C_{a,r}$ is the circle of centre a and radius r, and n is an integer. Show by direct computation that $I_n = 0$ for $n \neq -1$, and $I_n = 2\pi i$ for $n = -1$.

10. Consider the integral of the function $f(z) = e^{iz^2}$ round the closed path Γ in the complex z plane given in Fig. 3:

$$\oint_\Gamma e^{iz^2}\ dz.$$

(a) Evaluate this integral for arbitrary $R > 0$ in Fig. 3.
(b) Consider the integral of f along the straight-line segment joining the points $z = Re^{i\pi/4}$ and

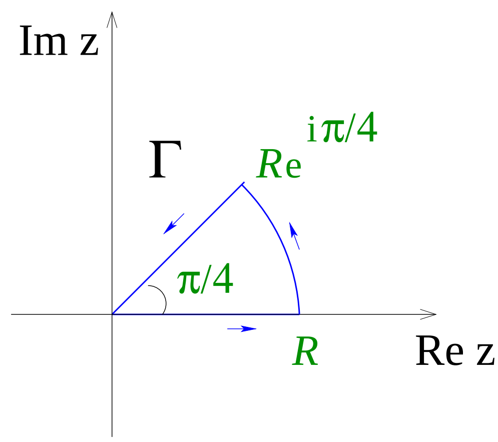

Fig.3

$z = 0$ in Fig. 3. Evaluate this integral for $R \to \infty$.
(c) Consider the integral of f along the circular arc in Fig. 3 between the points $z = R$ and $z = Re^{i\pi/4}$. Use the Darboux inequality to show that this integral vanishes for $R \to \infty$.
(d) Use the results in (a), (b), (c) to compute the real integrals

$$\int_0^\infty dx\ \cos x^2\ ,\qquad \int_0^\infty dx\ \sin x^2 \qquad \text{(Fresnel integrals)}.$$

11. Use Cauchy integral formulas to determine the value of

$$(a)\ \oint_\Gamma dz\ \frac{\cos z}{z(z^2+8)}\quad ,\qquad (b)\ \oint_\Gamma dz\ \frac{z}{2z+1}\quad ,$$

where Γ is a square with centre at the origin and sides of length 2.

12. Use Cauchy integral formulas to determine the value of

$$(a)\ \oint_\gamma dz\ \frac{e^z}{z^3}\quad ,\qquad (b)\ \oint_\gamma dz\ \frac{\cosh z}{z^4}\quad ,$$

where γ is the circle $|z-1| = 2$.

13. Show that if function f is holomorphic over the entire complex plane and is bounded (i.e., for any z, $|f(z)| \leq M$ for a real constant M) then f must be constant.
[Consider Cauchy integral formula for the first derivative of f. Apply Darboux inequality to it.]

14. Apply Poisson integral formula to determine a function $u(x,y)$ that is harmonic in the upper half plane,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\ ,\quad y > 0\ ,$$

and obeys the following Dirichlet boundary condition on the real axis:

$$u(x,0) = c(x)\ ,\quad \text{with}\ \ c(x) = 1\ \text{ for }\ x > 0\ ,\ \ c(x) = -1\ \text{ for }\ x < 0\ .$$

15. Use Gauss' mean value theorem to compute the following integrals:

$$(a) \int_0^{2\pi} \cos(\cos\theta + i\sin\theta)\, d\theta \ ,$$

$$(b) \int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta \ , \quad (c) \int_0^{2\pi} \sin(\cos\theta)\sinh(\sin\theta)\, d\theta \ .$$

16. Verify that the following functions are harmonic

$$(a)\ f_1(x,y) = e^{-2xy}\sin(x^2 - y^2) \qquad (b)\ f_2(x,y) = 2(1 + x^2 - y^2) + 3x^2y - y^3$$

and determine their integral round the circle in the xy plane with centre at the origin and radius 1.

17. Take the principal branch of the square root function $f(z) = \sqrt{z}$

$$\sqrt{z} = \sqrt{r}e^{i\theta/2}$$

defined with θ between 0 and 2π ($z = re^{i\theta}$) setting the branch cut from 0 to ∞ on the real positive semiaxis.
(a) Evaluate the integral of $\sqrt{z}$ on the circle $|z| = 1$.
[Suggestion: Consider the integral of $\sqrt{z}$ on the contour Γ in Fig. 4 and apply Cauchy theorem to this. Write down the relationship between the integral of $\sqrt{z}$ on Γ and the integral of $\sqrt{z}$ on the circle $|z| = 1$, taking the limit in which the radius ε of the inner circle in Fig. 4 goes to 0. Compute explicitly the contributions from the integrations along the straight-line segments above and below the branch cut.]

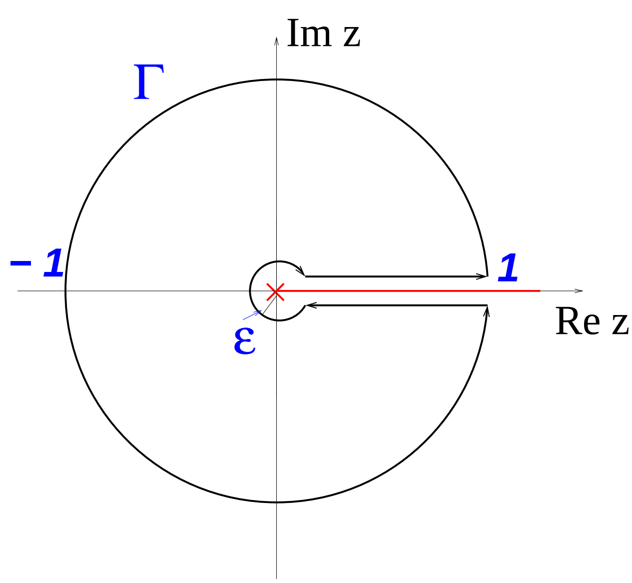

Fig.4

(b) Next consider the integral of $\sqrt{z}$ on any path in the upper half plane joining the points $z = -1$ and $z = 1$,

$$\int_{-1}^{1} \sqrt{z}\, dz \ ,$$

and the integral on any path in the lower half plane joining the same two points $z = -1$ and $z = 1$. Show that the results are different and are given respectively by $2(1+i)/3$ and $2(-1+i)/3$. Use this as a cross-check on the result of part (a).