Functions of a complex variable (S1)

Problem sheet 1

I. Complex numbers; elementary functions; sets in the complex plane

1. Determine all fifth roots of unity: $w = \sqrt[5]{1}$.
2. Determine all cubic roots of $z = i - 1$: $w = (i-1)^{1/3}$.
3. Solve the equation $(z+i)^5 + (z-i)^5 = 0$.
4. Find the real and imaginary parts of (a) $\cos i$ and (b) $\sin i$.
5. Verify that for any two complex numbers z_1 and z_2

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad , \qquad |z_1 + z_2| \geq |\,|z_1| - |z_2|\,| \qquad (\text{"triangle inequalities"}) \quad .$$

6. Solve the equations

(a) $e^z = -1$, (b) $e^z = -2$, (c) $\sinh z = 0$, (d) $\sin z = 3$.

7. Draw the curves in the complex z plane

(a) $|z-1| = 1$, (b) $(\mathrm{Re}\ z)^2 + 2\ (\mathrm{Im}\ z)^2 = 1$, (c) $\mathrm{Im}\ z^2 = 2$.

8. Find all the values of

(a) i^i , (b) $(1+i)^i$, (c) $(-1)^{1/\pi}$.

9. Find the real and imaginary parts of the principal branch of the logarithm of

(a) $z = 1 - i$, (b) $z = 1 + i$, (c) $z = i$.

10. Consider the set $S = \left\{ \frac{i}{n} : n = 1, 2, \ldots \right\}$ in the complex plane. (a) What are the limit points of S? (b) What are the interior and boundary points of S? (c) State whether or not S is open; closed; bounded; connected; compact.

II. Complex differentiation; holomorphic functions

11. Show that the function $f(z) = z^2$ is holomorphic in the entire complex plane, while the function $f(z) = |z|^2$ is holomorphic nowhere.
12. Determine the subsets of the complex z plane in which each of the following functions is holomorphic,

$$\text{(a)}\ \frac{1}{e^z - 1} \quad , \qquad \text{(b)}\ \frac{1}{(1+z^2)^2} \quad , \qquad \text{(c)}\ \cos \bar{z} \quad , \qquad \text{(d)}\ \tan z \quad ,$$

and calculate the derivative of each function in its region of holomorphy.

13. Let the function f be holomorphic on the domain D in $\mathbb{C}$.

    (a) Show that if f is real-valued then f must be constant.

    (b) Show that if $|f|$ is constant then f must be constant.

14. (a) Determine which of the following functions $u(x, y)$ are harmonic:

    (i) $x^2 - y^2 - x$, (ii) $\sin x \cosh y$, (iii) $e^{-x} \cos y + xy$, (iv) $\sin x - \cos y$, (v) $x - y$.

    (b) For each of the harmonic functions above, find a holomorphic function of which it is the real part, and find a harmonic conjugate function $v(x, y)$.

15. Take real-valued function u on domain D in $\mathbb{C}$, $u : D \to \mathbb{R}$. Let u be harmonic on D. Show that if u^2 is harmonic on D then u is constant.

16. (a) Find the holomorphic function whose imaginary part is given by

$$e^{-x}(x \cos y + y \sin y)$$

    and which vanishes at the origin.

    (b) Determine the family of curves in the xy plane which are orthogonal to the curves

$$e^{-x}(x \cos y + y \sin y) = \text{const.} \ .$$

17. Locate and classify the singular points of the following functions, and determine whether they are isolated:

$$\text{(a)} \ \frac{\cos z}{(z+i)^3} \ , \qquad \text{(b)} \ \frac{z^2}{(z^2-1)(z^2+4)} \ , \qquad \text{(c)} \ \frac{1}{\sin 1/z} \ .$$

18. Classify the behaviour of the following functions at $z = \infty$:

$$\text{(a)} \ z(1+z^2) \ , \qquad \text{(b)} \ e^z \ , \qquad \text{(c)} \ \frac{1+z^2}{z^2} \ .$$

19. Consider the mapping specified by the function

$$f : \ z \mapsto w = z + \frac{1}{z} \ .$$

    (a) Give the subset of the complex z plane in which f is holomorphic.

    (b) Determine whether the mapping is conformal in the region of holomorphy.

    (c) Onto which subset of the w plane does f map the upper half and lower half of circle $|z| = 1$?

    (d) What is the image through f of a circle $|z| = \rho$ with $\rho \neq 1$?

20. Consider the mapping specified by the function

$$f : \ z \mapsto w = \frac{i-z}{i+z} \ .$$

    (a) Is the mapping conformal?

    (b) What is the image through f of the real axis $z = x$?

    (c) Onto which subset of the w plane does f map the upper half plane Im $z > 0$?